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Diffusion Process on Time-Inhomogeneous Manifolds

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Abstract

Let $L_t := \Delta_t + Z_t$, $t \in [0, T_c)$ on a differential manifold equipped with time-depending complete Riemannian metric $(g_t)_{t\in[0,T_c)}$, where Δ_t is the Laplacian induced by g_t and $(Z_t)_{t\in[0,T_c)}$ is a family of $C^{1,1}$-vector fields. We first present some explicit criteria for the non-explosion of the diffusion processes generated by L_t; then establish the derivative formula for the associated semigroup; and finally, present a number of equivalent semigroup inequalities for the curvature lower bound condition, which include the gradient inequalities, transportation-cost inequalities, Harnack inequalities and functional inequalities for the diffusion semigroup.

Keywords: **Metric flow, curvature, coupling, transportation-cost inequality, Harnack inequality, L_t-diffusion process**

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1 Introduction

Let M be a d-dimensional differential manifold without boundary which carries a $C^{1,\infty}$-family of time-depending Riemannian metrics $(g_t)_{t\in[0,T_c)}$ for some $T_c \in (0, \infty]$. Let ∇^t be the Levi-Civita connection associated with the metric g_t, and Δ_t be the associated Laplace-Beltrami operator. For simplicity, we take the notations: for $X, Y \in TM$,

$$\mathrm{Ric}_t^Z(X,Y) := \mathrm{Ric}_t(X,Y) - \left\langle \nabla_X^t Z_t, Y \right\rangle_t$$
$$\mathcal{R}_t^Z(X,Y) := \mathrm{Ric}_t^Z(X,Y) - \frac{1}{2}\partial_t g_t(X,Y),$$

where Ric_t is the Ricci curvature tensor with respect to g_t, $(Z_t)_{t\in[0,T_c)}$ is a $C^{1,1}$-family of vector fields, and $\langle\cdot,\cdot\rangle_t := g_t(\cdot,\cdot)$. Consider the elliptic operator $L_t := \Delta_t + Z_t$. Let X_t be the inhomogeneous diffusion process generated by L_t (called L_t-diffusion process). Assume that X_t is non-explosive before T_c. In this paper, we want to clarify the connection between behavior of the distribution of the L_t-diffusion process, and the geometry of their underlying time-inhomogeneous space. The main work is to study this inhomogeneous diffusion process by using a new curvature condition, i.e the low bound of $\mathcal{R}_t^Z$. Compared with usual Bakry-Emery's curvature condition, it contains an additional term $\partial_t g_t$.

In the time-homogeneous case, many excellent scholars did deep research on the development of stochastic analysis on manifolds. In [8, 12, 25], the derivative formula of the diffusion

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semigroup, known as Bismut-Elworthy-Li formula, was given by constructing a damped gradient operator. Based on this formula, virous equivalent semigroup inequalities for the curvature lower bound (see e.g. [28, 7] and reference within) had been proved. All conclusions above were considered for the constant manifold without boundary. For the case with boundary, we refer the readers to [30, 31, 32, 33] for details.

Before moving on, let us briefly recall some known results in the time-inhomogeneous Riemannian setting. In [1], Coulibaly et al constructed the g_t-Brownian motion (i.e. the diffusion generated by $\frac{1}{2}\Delta_t$), established the Bismut formula when $(g_t)_{t\geq 0}$ is the Ricci flow, which in particular implies the gradient estimates of the associated heat semigroup. Next, by constructing horizontal diffusion processes, Coulibaly [2] investigated the optimal transportation inequality on time-inhomogeneous space. Moreover, Kuwada and Philipowski studied the non-explosion of g_t-Brownian motion in [17] for the super Ricci flow, and the first author developed the coupling method to estimate the gradient of the semigroup in [19]. Note that in [19], the coupling process was constructed as the limit of a sequence of time-inhomogeneous geodesic random walks, which avoids dealing with the cut-locus, but the coupling process is not direct and the proof is relatively complex. In this paper, we aim to give a intuitive construction of the coupling based on Wang's method (see [29, Chapter 3]). And it is further applied to transportation-cost inequalities and gradient estimation. For more development on the research on stochastic analysis on time-inhomogeneous space. See [18] for reviewing the monotonicity of $\mathcal{L}$-transportation cost from a probabilistic point; see [9, 10] for the stochastic analysis on path space over time-inhomogeneous space.

The rest parts of the paper are organized as follows. In Section 2, we first introduce the L_t-diffusion processes, and then present several explicit curvature conditions for the non-explosive of these processes. In Section 3, we first establish the derivative formula and derive the gradient estimates of diffusion semigroup, then characterize $\mathcal{R}_t^Z$ by using the formulae of the gradient of the semigroup. Finally, in Section 4, we present some equivalent inequalities of the semigroup for the lower bound of $\mathcal{R}_t^Z$.

2 The L_t-diffusion process

Let $\mathcal{F}(M)$ be the frame bundle over M and $\mathcal{O}_t(M)$ be the orthonormal frame bundle over M with respect to g_t. Let $\mathbf{p} : \mathcal{F}(M) \to M$ be the projection from $\mathcal{F}(M)$ onto M. Let $\{e_\alpha\}_{\alpha=1}^d$ be the canonical orthonormal basis of $\mathbb{R}^d$. For any $u \in \mathcal{F}(M)$, let $H_i^t(u)$ be the ∇^t horizontal lift of ue_i and $\{V_{\alpha,\beta}(u)\}_{\alpha,\beta=1}^d$ be the canonical basis of vertical fields over $\mathcal{F}(M)$, defined by $V_{\alpha,\beta}(u) = Tl_u(\exp(E_{\alpha,\beta}))$, where $E_{\alpha,\beta}$ is the canonical basis of $\mathcal{M}_d(\mathbb{R})$, the $d \times d$ matric space over $\mathbb{R}$, and $l_u : Gl_d(\mathbb{R}) \to \mathcal{F}(M)$ is the left multiplication from the general linear group to $\mathcal{F}(M)$, i.e. $l_u \exp(E_{\alpha,\beta}) = u \exp(E_{\alpha,\beta})$.

Let $B_t := (B_t^1, B_t^2, \cdots, B_t^d)$ be a $\mathbb{R}^d$-valued Brownian motion on a complete filtered probability space $(\Omega, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ with the natural filtration $\{\mathscr{F}_t\}_{t\geq 0}$. Assume the elliptic generator

L_t is a C^1 functional of time with associated metric g_t:

$$L_t = \Delta_t + Z_t$$

where Z_t is a $C^{1,1}$ vector field on M. As in the time-homogeneous case, to construct the L_t-diffusion process, we first construct the corresponding horizontal diffusion process generated by $\Delta_{\mathcal{O}_t(M)} + H_{Z_t}^t$ by solving the Stratonovich stochastic differential equation

$$\begin{cases} \mathrm{d}u_t = \sqrt{2}\sum_{i=1}^{d} H_i^t(u_t) \circ \mathrm{d}B_t^i + H_{Z_t}^t(u_t)\mathrm{d}t - \frac{1}{2}\sum_{\alpha,\beta=1}^{d} \partial_t g_t(u_t e_\alpha, u_t e_\beta) V_{\alpha\beta}(u_t)\mathrm{d}t, \\ u_0 \in \mathcal{O}_0(M), \end{cases}$$

where $\Delta_{\mathcal{O}_t(M)}$ is the horizontal Laplace operator on $\mathcal{O}_t(M)$; $H_{Z_t}^t(u_t)$ is ∇^t horizontal lift Z_t. Similarly as explained in [1], the last term promises $u_t \in \mathcal{O}_t(M)$. Since $\{H_{Z_t}^t\}_{t\in[0,T_c)}$ is $C^{1,1}$-smooth, the equation has a unique solution to its life time $\zeta := \lim_{n\to\infty} \zeta_n$, where

$$\zeta_n := \inf\{t \in [0, T_c) : \rho_t(\mathbf{p}u_0, \mathbf{p}u_t) \geq n\}, \ n \geq 1, \ \ \inf \varnothing := T_c,$$

where ρ_t stands for the distance induced by the metric g_t. Let $X_t = \mathbf{p}u_t$. Then X_t solves the equation

$$\mathrm{d}X_t = \sqrt{2}u_t \circ \mathrm{d}B_t + Z_t(X_t)\mathrm{d}t, \ X_0 = x := \mathbf{p}u_0$$

up to the life time ζ. By the Itô formula, for any $f \in C_0^2(M)$,

$$f(X_t) - f(x) - \int_0^t L_s f(X_s)ds = \sqrt{2}\int_0^t \left\langle u_s^{-1}\nabla^s f(X_s), \mathrm{d}B_s\right\rangle_s$$

is a martingale up to ζ; that is X_t the diffusion process generated by L_t, called the L_t-diffusion process. When $Z_t = 0$, then $\tilde{X}_t := X_{t/2}$ is generated by $\frac{1}{2}\Delta_t$ and is known as the g_t-Brownian motion.

Throughout the paper, we only consider the case where the L_t-diffusion process is non-explosive. In this case

$$P_{s,t}f(x) := \mathbb{E}(f(X_t)|X_s = x), \ x \in M, \ 0 \leq s \leq t < T_c, \ f \in \mathscr{B}_b(M)$$

gives rise to a Markov semigroup $\{P_{s,t}\}_{0\leq s\leq t<T_c}$ on $\mathscr{B}_b(M)$, which is called the diffusion semigroup generated by L_t. Here and in what follows, $\mathbb{E}^x$ (resp. $\mathbb{P}^x$) stands for the expectation (resp. probability) taken for the underlying process starting from point x. Fixed a certain point $o \in M$, denote $\rho_t(o, x)$ by $\rho_t(x)$ for simplicity.

2.1 The non-explosion

The main result in this subsection is presented as follows.

Theorem 2.1. *Let $\psi \in C(0,\infty)$ and $h \in C([0, T_c))$ be non-negative such that for any $t \in [0, T_c)$,*

$$\left(L_t\rho_t + \frac{\partial \rho_t}{\partial t}\right)(x) \leq h(t)\psi(\rho_t(x)) \tag{2.1}$$

holds outside $\mathrm{Cut}_t(o)$*, the cut-locus of* o *associated with* g_t*. If*

$$\int_1^\infty \mathrm{d}t \int_1^t \exp\left[-\int_r^t \psi(s)\mathrm{d}s\right]\mathrm{d}r = \infty, \tag{2.2}$$

then the diffusion process generated by L_t *is non-explosive.*

Proof. Fix $T \in (0, T_c)$, then there exists $c := \sup_{t\in[0,T]} h(t) > 0$,

$$(L_t\rho_t + \frac{\partial \rho_t}{\partial t})(x) \le c\psi(\rho_t(x)), \text{ and } t \in [0, T].$$

Let

$$f(x) = \int_1^x \mathrm{d}t \int_1^t \exp(-c\int_r^t \psi(s)\mathrm{d}s)\mathrm{d}r.$$

It is easy to see that $\lim_{x\to\infty} f(x) = \infty$ by Eq.(2.2). We know that for $t \in [0, T]$,

$$\begin{aligned}\left(L_t f \circ \rho_t + f' \circ \rho_t \frac{\partial \rho_t}{\partial t}\right)(x) &= \left(f' \circ \rho_t L_t \rho_t + f'' \circ \rho_t + f' \circ \rho_t \frac{\partial \rho_t}{\partial t}\right)(x) \\ &= \left(f'' \circ \rho_t + cf'(\rho_t)\psi(\rho_t)\right)(x) \le 1\end{aligned}$$

holds outside $\mathrm{Cut}_t(o)$. Then, by [17, Theorem 2], i.e. the Itô formula for radial part of X_t,

$$\begin{aligned}\mathrm{d}f \circ \rho_t(X_t) &\le \sqrt{2}\left\langle u_t^{-1}\nabla^t f \circ \rho_t(X_t), \mathrm{d}B_t\right\rangle_{\mathbb{R}^d} + \left(L_t f \circ \rho_t + f' \circ \rho_t \frac{\partial \rho_t}{\partial t}\right)(X_t)\mathrm{d}t \\ &\le \sqrt{2}\left\langle u_t^{-1}\nabla^t f \circ \rho_t(X_t), \mathrm{d}B_t\right\rangle_{\mathbb{R}^d} + \mathrm{d}t\end{aligned}$$

holds up to the life time ζ. In particular, if $X_0 = x \in M$, then

$$f(n)\mathbb{P}^x(\zeta_n \le t) \le \mathbb{E}^x f(X_{t\wedge\zeta_n}) \le f(\rho_0(x)) + t, \quad t \in [0, T).$$

Since $f(n) \to \infty$ as $n \to \infty$, this implies that

$$\mathbb{P}^x(\zeta \le t) \le \lim_{n\to\infty} \mathbb{P}^x(\zeta_n \le t) \le \lim_{n\to\infty} \frac{f(\rho_0(x)) + t}{f(n)} = 0, \quad t \in [0, T).$$

Therefore $\mathbb{P}(\zeta \ge T) = 1$. Since T is arbitrary, we have

$$\mathbb{P}(\zeta = T_c) = 1.$$

□

We remark here that very recently in [17], Kuwada and Philipowski have proved that the g_t-Brownian motion is non-explosive when the family of metrics evolves under backwards super Ricci flow.

As a consequence of Theorem 2.1, we present some explicit curvature conditions for the non-explosion of the L_t-diffusion process, which extend the corresponding known conditions in [13] for the constant metric case and [17] for the backwards super Ricci flow case. As usual, for any two-tensor $\mathbf{T}_t$, and any function f, we write $\mathbf{T}_t \ge f$, if $\mathbf{T}_t(X, X) \ge f\langle X, X\rangle_t$ holds for $X \in TM$. We have

Corollary 2.2. *The diffusion process X_t is non-explosive in each of the following situations:*

(1) *There exists a non-negative $\phi \in C([0,\infty))$ and $h \in C([0,T_c))$, such that $\mathcal{R}_t^Z \geq -h(t)\phi(\rho_t)$ and (2.2) holds for $\psi(s) := \int_0^s \phi(r)\mathrm{d}r$. In particular, it is the case if $\mathcal{R}_t^Z \geq -h(t)\log(e+\rho_t)$ holds.*

(2) *There exist non-negative and non-decreasing functions $\phi, \psi \in C(0,\infty)$ and $h \in C([0,T_c))$ such that (2.2) holds, $\mathrm{Ric}_t \geq -h(t)\phi(\rho(t,\cdot))$ and*

$$\begin{aligned} &\partial_t \rho_t + \langle Z_t, \nabla^t \rho_t \rangle_t + \sqrt{(d-1)\phi(\rho_t)}\coth\left(\sqrt{\phi(\rho_t)/(d-1)}\,\rho_t\right) \\ &\leq h(t)\psi(\rho_t). \end{aligned} \tag{2.3}$$

holds outside $\mathrm{Cut}_t(o)$. In particular, it is the case that if $\mathrm{Ric}_t \geq -h(t)(1+\rho_t^2)\log^2(e+\rho_t)$ and $\partial_t\rho_t + \langle Z_t, \nabla^t \rho_t\rangle_t \leq h(t)(1+\rho_t)\log(e+\rho_t)$ holds outside $\mathrm{Cut}_t(o)$.

Proof. (a) Let $x \notin \mathrm{Cut}_t(o)$ and $x \neq o$. Fix $t \in [0,T_c)$. Let γ be a minimizing unit-speed g_t-geodesic from o to x. Let $u := (u^1, u^2, \cdots, u^d) \in \mathcal{O}_t^x M$, the orthonormal basis of $T_x M$ w.r.t. g_t such that $u^d = \dot\gamma(\rho_t(x))$. Let $\{J_i\}_{i=1}^{d-1}$ be Jacobi fields along γ such that $J_i(t) = 0$ and $J_i(\rho_t(x)) = u^i$, $1 \leq i \leq d-1$. By the second variational formula, we have

$$\Delta_t \rho_t(x) = \sum_{i=1}^{d-1} \int_0^{\rho_t(x)} \left(|\nabla_{\dot\gamma}^t J_i|_t^2 - \langle R^t(\dot\gamma, J_i)\dot\gamma, J_i\rangle_t\right)(s)\mathrm{d}s.$$

Let U_i be the g_t-parallel vector field along γ such that $U_i(\rho_t(x)) = u^i$ and let $f(s) = 1 \wedge \frac{s}{\rho_t(x)\wedge 1}$. By the index lemma and noting that (see [22, Lemma 5 and Remark 6]),

$$\partial_t \rho_t(x) = \frac{1}{2}\int_0^{\rho_t(x)} \partial_t g_t(\dot\gamma(s), \dot\gamma(s))\mathrm{d}s, \tag{2.4}$$

we have

$$\begin{aligned} &L_t\rho_t(x) + \partial_t\rho_t(x) \\ \leq &\int_0^{\rho_t(x)} \left((d-1)f'^2 - f^2\mathrm{Ric}_t(\dot\gamma,\dot\gamma)\right)(\gamma(s))\mathrm{d}s \\ &+ \frac{1}{2}\int_0^{\rho_t(x)} \partial_t g_t(\dot\gamma(s),\dot\gamma(s))\mathrm{d}s + \langle Z_t, \nabla^t\rho_t\rangle_t(o) + \int_0^{\rho_t(x)} \left\langle \nabla^t_{\dot\gamma(s)} Z_t, \dot\gamma(s)\right\rangle_t \mathrm{d}s \\ \leq &\frac{d-1}{\rho_t(x)} + |Z_t(o)|_t + h(t)\int_0^{\rho_t(x)} \phi(s)\mathrm{d}s \\ \leq &(h(t) + 1 + |Z_t(o)|_t)\left(\frac{d-1}{\rho_t(x)} + 1 + \int_0^{\rho_t(x)}\phi(s)\mathrm{d}s\right) := \bar h(t)\bar\psi(\rho_t(x)). \end{aligned}$$

It is easy to see that (2.2) holds for $\bar\psi$ if and only if it holds for ψ. Then the desired assertion follows from Theorem 2.1.

(b) By the Laplacian comparison theorem and the lower curvature condition of Ric_t, one has

$$\Delta_t\rho_t \leq \sqrt{(d-1)\phi(\rho_t)}\coth\left(\sqrt{\phi(\rho_t)/(d-1)}\,\rho_t\right).$$

Therefore, (2.3) implies (2.1). By Theorem 2.1, the L_t-diffusion process is non-explosive. □

2.2 Kolmogrov equations

Theorem 2.3. *For any $f \in \mathscr{B}_b(M)$, the backward Kolmogorov equation*

$$\frac{\mathrm{d}}{\mathrm{d}s}P_{s,t}f = -L_s T_{s,t}f, \quad 0 \le s \le t < T_c \tag{2.5}$$

holds. If further $f \in C^2(M)$ such that $\|L_t f\|_\infty$ is locally bounded, then the forward Kolmogorov equation

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{s,t}f = P_{s,t}L_t f, \quad 0 \le s \le t < T_c \tag{2.6}$$

holds.

The proof is completely similar to the time-homogeneous case (see [34, Theorem 2.1.3]), and we thus omit it. See also [20] for the integration form: for any $0 \le s < t < T_c$,

$$P_{s,t}f = f + \int_s^t L_r P_{r,t} f \mathrm{d}r, \text{ for } f \in \mathscr{B}_b(M);$$
$$P_{s,t}f = f + \int_s^t P_{s,r} L_r f \mathrm{d}r, \text{ for } f \in C_0^2(M).$$

For $0 < t < T_c$, by Eq. (2.5), we know that $P_{s,t}f$ is the solution of the following problem

$$\begin{cases} \partial_s u(s,x) = -L_s u(t,x), \ s \in [0,t]; \\ u(t,x) = f(x). \end{cases} \tag{2.7}$$

On the other hand, for fixed time $T \in [0, T_c)$, let $(X_t^T)_{t\in[0,T]}$ be the $L_{(T-t)}$-diffusion process with semigroup $\{\overline{P}_{s,t}\}_{0\le s\le t\le T}$. Then $\overline{P}_{T-t,T}f$ solves the equation

$$\begin{cases} \partial_t u(t,x) = L_t u(t,x), t \in [0,T], \\ u(0,x) = f(x). \end{cases} \tag{2.8}$$

This time-reversed argument has important applications in the lifetime. For instance, Perelman used the reverse Ricci flow in his proof of the Poincaré conjecture; the Bismute type formula and gradient estimate has been investigated in [1] by using the reverse g_t-Brownian motion.

3 Formulae for $\nabla^s P_{s,t}$ and $\mathcal{R}_t^Z$

3.1 Bismut formula

The derivative formula for diffusion semigroup, known as Bismut-Elworthy-Li formula, is due to [8, 12], see Thalmaier [25] for a more general version. For the inhomogeneous case, Coulibaly et al. established in [1] the derivative formula for g_t-Brownian motion. Here, we will simplify the proof in [1] and present a more general version of the formula following the line of [25].

Let us introduce the $\mathbb{R}^d \otimes \mathbb{R}^d$-valued process $\{Q_{r,t}\}_{0<s\le t<T_c}$, which solves the ordinary differential equation

$$\frac{\mathrm{d}Q_{r,t}}{\mathrm{d}t} = -\mathcal{R}_t^Z(u_t)Q_{r,t}, \quad Q_{r,r} = I, \tag{3.1}$$

where u_t is the horizontal L_t-diffusion process with $\mathbf{p}u_0 = x$, and $\mathcal{R}_t^Z(u_t) \in \mathbb{R}^d \otimes \mathbb{R}^d$ satisfies

$$\left\langle \mathcal{R}_t^Z(u_t)a, b \right\rangle_{\mathbb{R}^d} = \mathcal{R}_t^Z(u_t a, u_t b), \quad a, b \in \mathbb{R}^d.$$

Let $K \in C([0, T_c) \times M)$ such that $\mathcal{R}_t^Z \geq K(t, \cdot),\ t \in [0, T_c)$. We have

$$\|Q_{r,t}\| \leq \exp\left[-\int_r^t K(s, X_s)\mathrm{d}s \right],$$

where $\|\cdot\|$ is the operator norm on $\mathbb{R}^d$. The following is our main result in this section. We denote $Q_t := Q_{0,t}$ for simplicity.

Theorem 3.1. *Let $0 \leq s < t < T_c$, $x \in M$ and D be a compact domain in $[s,t] \times M$ such that $(s,x) \in D^\circ$, the interior of D. Let $\tau_D := \inf\{r > s : (r, X_r) \notin D\}$, where $X_s = x$. Let $F \in C^2([s,t] \times D)$ satisfy the heat equation*

$$\partial_r F(r, \cdot) = -L_r F(r, \cdot), \quad r \in [s,t].$$

Then for any adapted absolutely continuous $\mathbb{R}_+$-valued process h such that $h(s) = 0,\ h(u) = 1$ for $u \geq t \wedge \tau_D$, and $\mathbb{E}(\int_s^t h'(u)^2 \mathrm{d}u)^\alpha < \infty$ for some $\alpha > \frac{1}{2}$,

$$(u_s)^{-1}\nabla^s F(s, \cdot)(x) = \frac{1}{\sqrt{2}}\mathbb{E}\left\{ F(t \wedge \tau_D, X_{t\wedge\tau_D}) \int_s^t h'(u) Q_{s,u}^* \mathrm{d}B_u \;\middle|\; X_s = x \right\}. \tag{3.2}$$

Proof. First, let $F_s = F(s, \cdot)$ for simplicity. By Theorem 3.1 and the well known Weitzenböck formula (see e.g.[15, 14]), we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}(\mathrm{d}F_s) &= -\mathrm{d}L_s F_s = -\mathrm{d}\{-\delta\mathrm{d}F_s + (\mathrm{d}F_s)(Z_s)\} \\ &= -\left[\Box_1(\mathrm{d}F_s) + \nabla^s_{Z_s}(\mathrm{d}F_s) + (\nabla^s Z_s)(\nabla^s F_s, \cdot)\right] \\ &= -\left(\Delta_s^1(\mathrm{d}F_s) + \nabla^s_{Z_s}(\mathrm{d}F_s)\right) + \mathrm{Ric}_s^Z(\cdot, \nabla^s F_s), \end{aligned} \tag{3.3}$$

where $\Box_1 = -\delta\mathrm{d} - \mathrm{d}\delta$ and $\Delta_s^1 = \mathrm{tr}(\nabla^s)^2$ is defined on Ω^1, the smooth section of one-form. On the other hand, for $f \in C^2(M)$, let $\widetilde{\mathrm{d}f}(u_t) \in \mathbb{R}^d$ such that

$$\widetilde{\mathrm{d}f}(u_t) = \sum_{i=1}^d u_t e_i(f) e_i, \text{ i.e. } \left\langle \widetilde{\mathrm{d}f}(u_t), a \right\rangle = \mathrm{d}f(u_t a), \quad a \in \mathbb{R}^d.$$

Then, by the Itô formula and the definition of Q_t, for $a \in \mathbb{R}^d$, we have

$$\begin{aligned} \mathrm{d}(\mathrm{d}f)(u_t Q_t a) &= \mathrm{d}\left\langle \widetilde{\mathrm{d}f}(u_t), Q_t a \right\rangle_{\mathbb{R}^d} = \left\langle \mathrm{d}\widetilde{\mathrm{d}f}(u_t), Q_t a \right\rangle + \left\langle \widetilde{\mathrm{d}f}(u_t)\mathrm{d}t, \mathrm{d}Q_t a \right\rangle \\ &= \left\langle H^t_{\sqrt{2}u_t\mathrm{d}B_t}\widetilde{\mathrm{d}f}(u_t), Q_t a \right\rangle + \left\langle L_{\mathcal{O}_t(M)}\widetilde{\mathrm{d}f}(u_t), Q_t a \right\rangle \mathrm{d}t \\ &\quad - \frac{1}{2}\sum_{i,j=1}^d \left\langle \partial_t g_t(u_t e_i, u_t e_j) V_{i,j}(u_t)\widetilde{\mathrm{d}f}(u_t), Q_t a \right\rangle \mathrm{d}t + \left\langle \widetilde{\mathrm{d}f}(u_t), \mathrm{d}Q_t a \right\rangle \\ &= \left\langle \widetilde{\nabla^t_{\sqrt{2}u_t\mathrm{d}B_t}\mathrm{d}f}, Q_t a \right\rangle + \left\langle \widetilde{\Delta_t \mathrm{d}f}, Q_t a \right\rangle \mathrm{d}t + \left\langle \widetilde{\nabla^t_{Z_t}\mathrm{d}f}, Q_t a \right\rangle \mathrm{d}t \\ &\quad - \frac{1}{2}\sum_{i,j=1}^d \partial_t g_t(u_t e_i, u_t e_j) \left\langle \mathrm{d}f(u_t e_j) e_i, Q_t a \right\rangle \mathrm{d}t + \left\langle \widetilde{\mathrm{d}f}(u_t), -\mathcal{R}_t^Z(u_t Q_t a) \right\rangle \mathrm{d}t \\ &= \nabla^t_{\sqrt{2}u_t\mathrm{d}B_t}(\mathrm{d}f)(u_t Q_t a) + \left[\Delta_t + \nabla^t_{Z_t}(\mathrm{d}f)\right](u_t Q_t a)\mathrm{d}t - \frac{1}{2}\partial_t g_t(\nabla^t f, u_t Q_t a)\mathrm{d}t \\ &\quad - \mathcal{R}_t^Z(\nabla^t f, u_t Q_t a)\mathrm{d}t \\ &= \left[\Delta_t + \nabla^t_{Z_t}(\mathrm{d}f)\right](u_t Q_t a)\mathrm{d}t + \nabla^t_{\sqrt{2}u_t\mathrm{d}B_t}(\mathrm{d}f)(u_t Q_t a) - \mathrm{Ric}_t^Z(\nabla^t f, u_t Q_t a)\mathrm{d}t, \end{aligned}$$

where the forth equality can be found in e.g. [13, Proposition 2.2.1],

$$H_Y^t \widetilde{\mathrm{d}f}(u_t) = \widetilde{\nabla_Y^t \mathrm{d}f}(u_t), \quad \text{and} \quad V_{i,j}(u_t)\widetilde{\mathrm{d}f}(u_t) = \mathrm{d}f(ue_j)e_i, \ Y \in T_{\mathbf{p}u_t}M$$

which can be easily checked by the definition of $V_{i,j}(u_t)$. Combining this with (3.3), we obtain

$$\mathrm{d}\left\langle \nabla^s F_s(X_s), u_s Q_s a\right\rangle_s = \sqrt{2}\mathrm{Hess}^s_{F_s}(u_s \mathrm{d}B_s, u_s Q_s a) \tag{3.4}$$

is a local martingale. Moreover, by the Itô formula

$$\mathrm{d}F_s(X_s) = \sqrt{2}\left\langle \nabla^s F_s(X_s), u_s \mathrm{d}B_s\right\rangle_s.$$

So that

$$F(t\wedge\tau_D, X_{t\wedge\tau_D}) = F(s,x) + \sqrt{2}\int_s^{t\wedge\tau_D} \left\langle \nabla^r F_r(X_r), u_r \mathrm{d}B_r\right\rangle_r.$$

Therefore,

$$\begin{aligned}
&\frac{1}{\sqrt{2}}\mathbb{E}^x\left\{F(t\wedge\tau_D, X_{t\wedge\tau_D})\int_s^t \left\langle h'(r)Q_r a, \mathrm{d}B_r\right\rangle_{\mathbb{R}}\right\}\\
=&\frac{1}{\sqrt{2}}\mathbb{E}^x\left\{\left(F(s,x)+\sqrt{2}\int_s^{t\wedge\tau_D}\left\langle \nabla^r F_r(X_r), u_r\mathrm{d}B_r\right\rangle_r\right)\int_s^t \left\langle h'(r)Q_r a, \mathrm{d}B_r\right\rangle_{\mathbb{R}}\right\}\\
=&\mathbb{E}^x\left\{\int_s^{t\wedge\tau_D}\left\langle \nabla^r F_r(X_r), u_r Q_r a\right\rangle_r (h-1)'(r)\mathrm{d}r\right\}\\
=&\mathbb{E}^x\left\{\left[\left\langle \nabla^r F_r(X_r), u_r Q_r a\right\rangle_r \cdot (h-1)(r)\right]\Big|_s^{t\wedge\tau_D}\right\}\\
&-\mathbb{E}^x\int_s^{t\wedge\tau_D}(h-1)(r)\mathrm{d}\left\langle \nabla^r F_r(X_r), u_r Q_r a\right\rangle_r\\
=&\left\langle \nabla^s F_s(x), u_s a\right\rangle_s,
\end{aligned}$$

where the last step follows from $(h-1)\mathrm{d}\left\langle \nabla^r F_r(X_r), u_r Q_r a\right\rangle_r$ is a martingale for $r\in[s,t]$ according to (3.4). □

If we choose some explicit process h in Theorem 3.1, then the associated gradient estimates of $P_{s,t}f$ can be achieved by only using local geometry of the manifold. It is easy to see when $F(t,x)=f(x)$, we have $F(r,x)=P_{r,t}f(x),\ r\in[s,t]$.

Corollary 3.2. *Let $\mathcal{R}_s^Z \geq K(s,\cdot)$ for some $K\in C([0,T_c)\times M)$. For any $x\in M$, let $\kappa_s(x) = \sup_{r\in[s,s+1]}(\sup_{B_r(x,1)} K(r,\cdot)^- + |Z_r|_r(x))$. Then there exists a constant $c>0$ such that*

$$|\nabla^s P_{s,t}f|_s \leq \frac{\|f\|_\infty \exp\left[c(1+\kappa_s)\right]}{\sqrt{(t-s)\wedge 1}}.$$

Proof. The idea is essentially due to [26]. Without loss generality, we only consider $s=0$ for simplicity. By the semigroup property and the contraction of $P_{s,t}$, it suffices to prove for $0<t\leq 1\wedge T_c$. We will apply the derivative formula to $D:=\{(r,y)\in[0,t]\times M : \rho_r(x,y)\leq 1\}$. D is closed and hence compact, since $\rho_t(x,y)$ is continuous as a function of (t,x,y). Let $f(t,X_t):=\cos(\pi\rho_t(x,X_t)/2)$. Let $X_0=x$ and

$$T(r) = \left(\int_0^r f^{-2}(u,X_u)\mathrm{d}u\right)\mathbf{1}_{\{r\leq\tau_D\}} + \infty\mathbf{1}_{\{r>\tau_D\}}.$$

Recall that τ_D is the first hitting time of (r, X_r) to ∂D. Let

$$\tau(r) = \inf\{u \geq 0 : T(u) \geq r\}, \quad r \geq 0.$$

Then $\tau \circ T(r) = T \circ \tau(r) = r$ for $r \leq \tau_D$. Since $f \leq 1$ and $\tau(r) \leq r$. Moreover,

$$\tau'(r) = \frac{1}{T' \circ \tau(r)} = f^2(\tau_{(r)}, X_{\tau(r)}).$$

Define $h(r) = 1 - \frac{1}{t}\int_0^{r\wedge\tau(t)} f^{-2}(r, X_r)\mathrm{d}r$. Then h meets the requirement of Theorem 3.1 and

$$\begin{aligned}\int_0^{\tau(t)} h'(r)^2 \mathrm{d}r &= \frac{1}{t^2}\int_0^{\tau(t)} f^{-4}(r, X_r)\mathrm{d}r = \frac{1}{t^2}\int_0^{\tau(t)} f^{-2}(r, X_r)\mathrm{d}T(r)\\ &= \frac{1}{t^2}\int_0^t f^{-2}(\tau(r), X_{\tau(r)})\mathrm{d}r. \end{aligned} \tag{3.5}$$

It is easy to see that $(\tau(r), X_{\tau(r)})$ is non-explosive on D. So it follows from Kendall's Itô formula that

$$\mathrm{d}f^{-2}(\tau(r), X_{\tau(r)}) \leq \mathrm{d}M_r + \left[f^2\left(L_{\tau(r)} + \partial_1\right)f^{-2}\right](\tau(r), X_{\tau(r)})\mathrm{d}r \tag{3.6}$$

holds for some local martingale M_r. By the comparison theorem and the definition of κ, there exists a constant $c_1 > 0$ such that

$$\sin(\pi\rho_r(x,\cdot)/2)\left(L_r + \partial_r\right)\rho_r(x,\cdot) \leq c_1(1 + \kappa_0(x)), \ r \in [0, t]$$

holds on D. Thus, there exists a constant $c_2 > 0$ such that

$$\begin{aligned} f^2\left(L_r + \partial_r\right)f^{-2} &= -2f^{-1}\left(L_r + \partial_r\right)f + 6f^{-2}|\nabla^r f|_r^2\\ &\leq c_2(1 + \kappa_0(x))f^{-2}, \quad r \in [0, t] \end{aligned} \tag{3.7}$$

holds on D. Combining this with (3.5) and (3.6), we obtain

$$\begin{aligned}\mathbb{E}^x\int_0^{\tau(t)} h'(r)^2\mathrm{d}r &\leq \frac{1}{t^2}\int_0^t \mathbb{E}^x f^{-2}(\tau(r), X_{\tau(r)})\mathrm{d}r \leq \frac{1}{t^2}\int_0^t e^{c_2(1+\kappa_0(x))r}\mathrm{d}r\\ &\leq \frac{c_3}{t}e^{c_3(1+\kappa_0(x))}, \ t \in (0, 1] \end{aligned} \tag{3.8}$$

for some constant $c_3 > 0$. Let $v \in T_xM$ and $|v|_0 = 1$. By the definition of Q_r and $\mathcal{R}_r^Z \geq -\kappa_0(x)$ on D, we have

$$|u_r Q_r u_0^{-1} v|_r \leq |v|_0 e^{c\kappa_0(x)}, \quad r \leq \tau(t), \quad 0 < t \leq 1.$$

Then, it follows from Theorem 3.1 and (3.8) that

$$\left|\left\langle\nabla^0 P_{0,t}f(x), v\right\rangle_0\right| \leq \|f\|_\infty e^{c\kappa_0(x)}\left(\mathbb{E}\int_0^{\tau(t)} h'(r)^2\mathrm{d}r\right)^{1/2} \leq \frac{\|f\|_\infty c_4 e^{c_4(1+\kappa_0(x))}}{\sqrt{t}}$$

holds for some constant $c_4 > 0$ and all $t \in (0, 1]$. This completes the proof.

□

Next, we present derivative formulae of $P_{s,t}$ without using hitting times.

Theorem 3.3. *Assume that* $(L_s+\partial_s)\rho_s^2 \le c+h_1(s)+h_2(s)\rho_s^2,\ s\in[0,T_c)$ *holds outside* $\mathrm{Cut}_s(o)$ *for some constant* $c>0$ *and some non-negative functions* $h_1,h_2\in C([0,T_c))$*. If*

$$\mathcal{R}_s^Z \ge h_3(s) - 16e^{-\int_0^s (h_2(u)+16)\mathrm{d}u}\rho_s^2 \tag{3.9}$$

holds for some $h_3\in C([0,T_c))$*, then for any* $0\le s\le t<T_c$*, and* $h\in C^1([s,t])$ *such that* $h(s)=0, h(t)=1$,

$$\begin{aligned} u_s^{-1}\nabla^s P_{s,t}f(x) &= \mathbb{E}\left\{Q_{s,t}^* u_t^{-1}\nabla^t f(X_t)\big| X_s=x\right\} \\ &= \frac{1}{\sqrt{2}}\mathbb{E}\left\{f(X_t)\int_s^t h'(r)Q_{s,r}^*\mathrm{d}B_r\middle| X_s=x\right\} \end{aligned} \tag{3.10}$$

holds for some $f\in C_b^1(M), x\in M$*. In particular, taking* $h(s)=\frac{(r-s)\wedge(t-s)}{t-s}$*, there holds*

$$u_s^{-1}\nabla^s P_{s,t}f(x) = \frac{1}{\sqrt{2}(t-s)}\mathbb{E}^x\left\{f(X_t)\int_s^t Q_{s,r}^*\mathrm{d}B_r\middle| X_s=x\right\}.$$

Proof. We again assume $s=0$. By the Itô formula (see [17, Theorem 2]),

$$\mathrm{d}\rho_s^2(X_s) \le 2\sqrt{2}\rho_s(X_s)\mathrm{d}b_s + (c+h_1(s)+h_2(s)\rho_s^2(X_s))\mathrm{d}s$$

holds for some one-dimensional Brownian motion b_t. Let

$$\lambda(s) = \int_0^s (h_2(u)+16)\mathrm{d}u.$$

Then we have

$$\begin{aligned} \mathrm{d}\left[e^{-\lambda(s)}\rho_s^2(X_s)\right] \le & e^{-\lambda(s)}\left[2\sqrt{2}\rho_s(X_s)\mathrm{d}b_s + \left(c+h_1(s)+h_2(s)\rho_s^2(X_s)\right)\mathrm{d}s\right] \\ & - e^{-\lambda(s)}(h_2(s)+16)\rho_s^2(X_s)\mathrm{d}s \\ =& 2\sqrt{2}e^{-\lambda(s)}\rho_s(X_s)\mathrm{d}b_s - 16e^{-\lambda(s)}\rho_s^2(X_s)\mathrm{d}s \\ & + (c+h_1(s))e^{-\lambda(s)}\mathrm{d}s. \end{aligned}$$

Therefore, letting $C(t,x) := e^{\rho_o^2(x)+ct+\int_0^t h_1(s)\mathrm{d}s}$, we have

$$\begin{aligned} \mathbb{E}^x \exp\left\{16\int_0^{t\wedge\zeta_n} \rho_s(X_s)e^{-\lambda(s)}\mathrm{d}s\right\} \le & \mathbb{E}^x \exp\left\{2\sqrt{2}\int_0^{t\wedge\zeta_n} \rho_s^2(X_s)e^{-\lambda(s)}\mathrm{d}b_s\right\}\cdot C(t,x) \\ \le & \mathbb{E}^x \exp\left\{16\int_0^{t\wedge\zeta_n} \rho_s^2(X_s)e^{-2\lambda(s)}\mathrm{d}s\right\}^{1/2}\cdot C(t,x). \end{aligned}$$

Thus

$$\mathbb{E}^x \exp\left\{16\int_0^{t\wedge\zeta_n} \rho_s^2(X_s)e^{-\lambda(s)}\mathrm{d}s\right\} \le C(t,x)^2.$$

Letting $n\to\infty$, we arrive at

$$\mathbb{E}^x \exp\left\{16\int_0^{t} \rho_s^2(X_s)e^{-\lambda(s)}\mathrm{d}s\right\} \le C(t,x)^2.$$

Combining this with (3.9), we conclude that for $K(s,x)=h_3(s)-16e^{-\int_0^s(h_2(u)+16)\mathrm{d}u}\rho_s^2$, one has

$$\mathcal{R}_s^Z \ge K(s,\cdot),\quad s\in[0,t],$$

and

$$\sup_{x\in\mathbf{K}}\mathbb{E}^x e^{\int_0^t K^-(s,X_s)\mathrm{d}s}=\sup_{x\in\mathbf{K}}\mathbb{E}^x\exp\left\{16\int_0^t e^{-\int_0^s(h_2(u)+16)\mathrm{d}u}\rho^2(s,X_s)\mathrm{d}s\right\}<\infty,$$

where $\mathbf{K}\subset M$ is a compact subset. Following the proof of [21, Theorems 3.1 and 9.1], we conclude that $\sup_{s\in[0,t]}\|\nabla^s P_{s,t}f\|_\infty<\infty$. Then the first equality follows from (3.10) by taking

$$\mathbb{E}^x\sup_{s\in[0,t]}|\langle\nabla^s P_{s,t}f(X_s),u_sQ_sa\rangle_s|\le\|\nabla^t f\|_\infty\mathbb{E}^x e^{\int_0^t K^-(s,X_s)\mathrm{d}s}<\infty,\ a\in\mathbb{R}^d\text{ and }\|a\|=1.$$

Thus

$$\langle\nabla^s P_{s,t}f(x),u_sQ_sa\rangle_s,\ s\in[0,t]$$

is a uniformly integrable martingale, and thus (3.2) holds for t in place of $t\wedge\tau_D$ and any $h\in C^1([0,t])$ with $h(0)=0,\ h(t)=1$. Therefore, the second equality holds. □

By Corollary 2.2 and the Laplacian comparison theorem, the assumption $(L_s+\partial_s)\,\rho_s^2\le c+h_1(s)+h_2(s)\rho_s^2$ in Theorem 3.3 follows from each of the following conditions:

(A1) there exists a non-negative $C\in C([0,T_c))$ such that $\mathcal{R}_t^Z\ge -C(t)$;

(A2) there exists two non-negative functions $C_1,C_2\in C([0,T_c))$, such that

$$\mathrm{Ric}_t\ge -C_1(t)(1+\rho_t^2),\ \text{and } \partial_t\rho_t+\left\langle Z_t,\nabla^t\rho_t\right\rangle_t\le C_2(t)(1+\rho_t).$$

3.2 Asymptotic Formula for $\mathcal{R}_t^Z$

In this subsection, we present the characterizations of $\mathcal{R}_t^Z$ by using the gradient of $P_{s,t}$, which is a extension of [24, 5] for the case with constant metric.

Theorem 3.4. *Let $x\in M$, for any $s\in[0,T_c)$, $X\in T_xM$, with $|X|_s=1$. Let $f\in C_0^\infty(M)$ such that $\nabla^s f(x)=X$ and $\mathrm{Hess}_f^s(x)=0$, and let $f_n=n+f$ for $n\ge1$. Then,*
(1) *for any $p>0$,*

$$\mathcal{R}_s^Z(X,X)=\lim_{t\downarrow s}\frac{P_{s,t}|\nabla^t f|_t^p(x)-|\nabla^s P_{s,t}f|_s^p(x)}{p(t-s)};\tag{3.11}$$

(2) *for any $p>1$,*

$$\begin{aligned}\mathcal{R}_s^Z(X,X)&=\lim_{n\to\infty}\lim_{t\downarrow s}\frac{1}{t-s}\left(\frac{p\{P_{s,t}f_n^2-(P_{s,t}f_n^{\frac{2}{p}})^p\}}{4(p-1)(t-s)}-|\nabla^s P_{s,t}f_n|_s^2\right)(x)\\&=\lim_{n\to\infty}\lim_{t\downarrow s}\frac{1}{t-s}\left(P_{s,t}|\nabla^t f|_t^2-\frac{p\{P_{s,t}f_n^2-(P_{s,t}f_n^{\frac{2}{p}})^p\}}{4(p-1)(t-s)}\right)(x);\end{aligned}\tag{3.12}$$

(3) *$\mathcal{R}_s^Z(X,X)$ is equal to each of the following limits:*

$$\lim_{n\to\infty}\lim_{t\downarrow s}\frac{1}{(t-s)^2}\left\{(P_{s,t}f_n)\left[P_{s,t}(f_n\log f_n)-(P_{s,t}f_n)\log P_{s,t}f_n\right]-(t-s)|\nabla^s P_{s,t}f|_s^2\right\}(x).\tag{3.13}$$

$$\lim_{n\to\infty}\lim_{t\downarrow t}\frac{1}{4(t-s)^2}\left\{4(t-s)P_{s,t}|\nabla^t f|_t^2+(P_{s,t}f_n^2)\log P_{s,t}f_n^2-P_{s,t}f_n^2\log f_n^2\right\}(x).\tag{3.14}$$

Proof. (1) Without loss generality, we only prove for $s=0$. The proof is similar to the corresponding ones in [24] for constant metric case. Since $\nabla^0 f = X$ and $\mathrm{Hess}_f^0(x)=0$. By the Bochner-Weitzenböck formula, we have

$$\Gamma_2^0(f,f)(x) := \frac{1}{2}L_0|\nabla^0 f|_0^2(x) - \left\langle \nabla^0 f, \nabla^0 L_0 f\right\rangle_0(x) = \mathrm{Ric}_0^Z(X,X).$$

Therefore, the first assertion follows from Theorem 2.3 and the Taylor expansion at point x (recall that $\mathrm{Hess}_f^0(x)=0$). That is

$$\begin{aligned}
&P_{0,t}|\nabla^t f|_t^p\\
=&|\nabla^0 f|_0^p + \left(L_0|\nabla^0 f|_0^p + \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}|\nabla^t f|_t^p\right)t + o(t)\\
=&|\nabla^0 f|_0^p + \left(\frac{p}{2}|\nabla^0 f|_0^{p-2}L_0|\nabla^0 f|_0^2 - \frac{p}{2}|\nabla^0 f|_0^{p-2}\partial_t g_t|_{t=0}(\nabla^0 f,\nabla^0 f)\right)t + o(t),
\end{aligned}$$

and

$$|\nabla^0 P_{0,t}f|_0^p = |\nabla^0 f|_0^p + pt|\nabla^0 f|_0^{p-2}\left\langle \nabla^0 L_0 f, \nabla^0 f\right\rangle_0 + o(t),$$

where in the first equality, we use the formula

$$\partial_t|\nabla^t f|_t^2 = -\partial_t g_t(\nabla^t f, \nabla^t f).$$

These equalities futher imply

$$\mathcal{R}_0^Z(X,X) = \lim_{t\to 0}\frac{P_{0,t}|\nabla^t f|_t^p(x) - |\nabla^0 P_{0,t}f|_0^p(x)}{pt}. \tag{3.15}$$

(2) Let $f_n = n+f$, which is positive for large n. We have, for small $t>0$ and large n,

$$\begin{aligned}
\frac{\mathrm{d}P_{0,t}f_n^2}{\mathrm{d}t} &= P_{0,t}L_t f_n^2 = P_{0,t}(2f_n L_t f_n + 2|\nabla^t f_n|_t^2);\\
\frac{\mathrm{d}^2 P_{0,t}f_n^2}{\mathrm{d}t^2} &= P_{0,t}L_t^2 f_n^2 + P_{0,t}\left(2f_n\frac{\mathrm{d}L_t f_n}{\mathrm{d}t} - 2\partial_t g_t(\nabla^t f_n, \nabla^t f_n)\right),
\end{aligned}$$

and

$$\begin{aligned}
\frac{\mathrm{d}(P_{0,t}f_n^{2/p})^p}{\mathrm{d}t} &= p(P_{0,t}f_n^{2/p})^{p-1}P_{0,t}L_t f_n^{2/p}\\
&= p(P_{0,t}f_n^{2/p})^{p-2}P_{0,t}\left(\frac{2}{p}f_n^{\frac{2}{p}-1}L_t f_n + \frac{2}{p}(\frac{2}{p}-1)f_n^{\frac{2}{p}-2}|\nabla^t f_n|_t\right),\\
\frac{\mathrm{d}^2(P_{0,t}f_n^{2/p})^p}{\mathrm{d}t^2} &= p(p-1)(P_{0,t}f_n^{2/p})^{p-2}(P_{0,t}L_t f_n^{2/p})^2 + p(P_{0,t}f_n^{2/p})^{p-1}P_{0,t}\bigg(L_t^2 f_n^{2/p}\\
&\quad + \frac{2}{p}f_n^{2/p-1}\frac{\mathrm{d}L_t f_n}{\mathrm{d}t} - \frac{2}{p}(\frac{1}{p}-1)\partial_t g_t(\nabla^t f_n, \nabla^t f_n)\bigg).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
P_{0,t}f_n^2 - (P_{0,t}f_n^{2/p})^p &= t\left(L_0 f_n^2 - pf_n^{\frac{2(p-1)}{p}}L_0 f_n^{2/p}\right) + \frac{t^2}{2}\bigg(L_0^2 f_n^2 - p(p-1)f_n^{\frac{2(p-2)}{p}}(L_0 f_n^{2/p})^2\\
&\quad - pf_n^{\frac{2(p-1)}{p}}L_0^2 f_n^{2/p} - 4\frac{(p-1)}{p}\partial_t g_t|_{t=0}(\nabla^0 f, \nabla^0 f)\bigg) + o(t^2)\\
&= \frac{8(p-1)t^2}{p}\left\langle \nabla^0 f, \nabla^0 L_0 f\right\rangle_0 - \frac{2(p-1)t^2}{p}\partial_t g_t|_{t=0}(\nabla^0 f, \nabla^0 f)\\
&\quad + \frac{4(p-1)t}{p}|\nabla^0 f|_0^2 + \frac{4(p-1)t^2}{p}\Gamma_2^0(f,f) + t^2 O(n^{-1}) + o(t^2).
\end{aligned}$$

We remark here that $o(t^2)$ depends on n but $O(n^{-1})$ is independent of t. By the definition of Γ_2^0 and

$$|\nabla^0 P_{0,t} f_n|_0^2 = |\nabla^0 f|_0^2 + 2t \left\langle \nabla^0 f, \nabla^0 L_0 f \right\rangle_0 + o(t), \tag{3.16}$$

we obtain the first equality in (2). And the second equality follows as

$$P_{0,t}|\nabla^t f|_t^2 = |\nabla^0 f|_0^2 + t\left(L_0|\nabla^0 f|_0^2 - \partial_t g_t|_{t=0}(\nabla^0 f, \nabla^0 f)\right) + o(t). \tag{3.17}$$

(3) The two equality in (3) can be proved by combining (3.16) and (3.17) with the following two asymptotic formulae respectively.

$$\begin{aligned}
&(P_{0,t}f_n)\{P_{0,t}(f_n\log f_n) - (P_{0,t}f_n)\log P_{0,t}f_n\}\\
=&(f_n + tO(1) + o(t))\Big\{t[L_0(f_n\log f_n) - (1+\log f_n)L_0 f_n]\\
&+\frac{t^2}{2}\left[L_0^2(f_n\log f_n) - (1+\log f_n)L_0^2 f_n - \frac{1}{f_n}(L_0 f_n)^2 - \frac{1}{f_n}\partial_t g_t|_{t=0}(\nabla^0 f, \nabla^0 f)\right] + o(t^2)\Big\}\\
=&t|\nabla^0 f|_0^2 + t^2\Gamma_2^0(f,f) + 2t^2\left\langle \nabla^0 f, \nabla^0 L_0 f\right\rangle_0 - \frac{1}{2}t^2\partial_t g_t|_{t=0}(\nabla^0 f, \nabla^0 f)\\
&+t^2 O(n^{-2}) + o(t^2),
\end{aligned}$$

and

$$\begin{aligned}
&(P_{0,t}f_n^2)\log P_{0,t}f_n^2 - P_{0,t}(f_n^2\log f_n^2)\\
=&t\left[(1+\log f_n^2)L_0 f_n^2 - L_0(f_n^2\log f_n^2)\right]\\
&+\frac{t^2}{2}\left[f_n^{-2}(L_0 f_n^2)^2 + (1+\log f_n^2)L_0^2 f_n^2 - L_0^2(f_n^2\log f_n^2) + 4\partial_t g_t|_{t=0}(\nabla^0 f_n, \nabla^0 f_n)\right] + o(t^2)\\
=&-4t|\nabla^0 f|_0^2 - 4t^2\left\langle \nabla^0 L_0 f, \nabla^0 f\right\rangle_0 + 2t^2\partial_t g_t|_{t=0}(\nabla^0 f, \nabla^0 f)\\
&-2t^2 L_0|\nabla^0 f|_0^2 + o(t^2) + t^2 O(n^{-1}).
\end{aligned}$$

□

4 Equivalent semigroup inequalities for the lower bound of $\mathcal{R}_t^Z$

In this section, we aim to provide various equivalent inhomogeneous semigroup properties for the curvature lower bound condition,

$$\mathcal{R}_t^Z \geq K(t,\cdot), \quad t \in [0, T_c). \tag{4.1}$$

In §4.1, we present some equivalent gradient inequities for (4.1). To derive more equivalent inequalities, we introduce in §4.2 two crucial couplings of the L_t-diffusion process. Finally, in §4.3, we present more equivalent semigroup inequities for (4.1).

4.1 Equivalent gradient inequlities

Theorem 4.1. *Assume* (**A1**) *or* (**A2**) *holds. Let $p \geq 1$ and $\tilde{p} = p \wedge 2$. Then for any $K \in C([0,T_c) \times M)$ such that $K(t,x)^-/\rho_t^2(x) \to 0$ as $\rho_t(x) \to \infty$, the following statements are equivalent each other:*

(1) (4.1) *holds.*

(2) *For any* $x \in M$, $0 \le s \le t < T_c$, *and* $f \in C_b^1(M)$,

$$|\nabla^s P_{s,t} f(x)|_s^p \le \mathbb{E}\{|\nabla^t f|_t^p(X_t)\exp[-p\int_s^t K(r, X_r)\mathrm{d}r]|X_s = x\}$$

(3) *For any* $0 \le s \le t < T_c$, $x \in M$ *and positive* $f \in C_b^1(M)$,

$$\frac{\tilde{p}[P_{s,t}f^2 - (P_{s,t}f^{1/\tilde{p}})^{\tilde{p}}]}{4(\tilde{p}-1)} \le \mathbb{E}\left\{|\nabla^t f|_t^2(X_t)\int_s^t e^{-2\int_u^t K(r,X_r)\mathrm{d}r}\mathrm{d}u\middle|X_s = x\right\},$$

where when $p = 1$, *the inequality is understood as its limit as* $p \downarrow 1$:

$$\begin{aligned}&P_{s,t}(f^2\log f^2)(x) - (P_{s,t}f^2\log P_{s,t}f^2)(x)\\ &\le 4\mathbb{E}\left\{|\nabla^t f|_t(X_t)\int_s^t e^{-2\int_u^t K(r,X_r)\mathrm{d}r}\mathrm{d}u|X_s = x\right\}.\end{aligned}$$

(4) *For any* $0 \le s \le t < T_c$, $x \in M$ *and positive* $f \in C_b^1(M)$,

$$\begin{aligned}&|\nabla^s P_{s,t}f|_s^2(x)\\ &\le \frac{[P_{s,t}f^{\tilde{p}} - (P_{s,t}f)^{\tilde{p}}](x)}{\tilde{p}(\tilde{p}-1)\int_s^t\left(\mathbb{E}\{(P_{u,t}f)^{2-\tilde{p}}(X_u)\exp\left[-2\int_s^u K(r,X_r)\mathrm{d}r\right]|X_s = x\}\right)^{-1}\mathrm{d}u},\end{aligned}$$

where when $p = 1$, *the inequality is understood as its limit as* $p \downarrow 1$:

$$|\nabla^s P_{s,t}f|_s^2(x) \le \frac{[P_{s,t}(f\log f) - (P_{s,t}f)\log P_{s,t}f](x)}{\int_s^t\left(\mathbb{E}\{P_{u,t}f(X_u)\exp\left[-2\int_s^u K(r,X_r)\mathrm{d}r\right]|X_s = x\}\right)^{-1}\mathrm{d}u}.$$

Proof. According to the proof of Theorem 3.3, $\mathbb{E}\exp(p\int_0^t K^-(s, X_s)\mathrm{d}s) < \infty$ holds for any $p > 0$, $0 \le t < T_c$ and $x \in M$. So according to Theorem 3.4, we obtain (1) by applying (2) to $f \in C_0^\infty(M)$ such that $\mathrm{Hess}_f^s(x) = 0$ or apply (3) to $n + f$ in place of f, or applying (4) to $(f+n)^{2/p}$ when $p > 1$ (resp. $f + n$ when $f + n$ when $p = 1$) in place of f. So, it suffices to show that (1) implies (2)–(4).

First, if $\mathcal{R}_t^Z \ge K(t,\cdot)$, $t \in [0, T_c)$, then by the first equality in (3.10) and (3.1), we have

$$\begin{aligned}|\nabla^s P_{s,t}f|_s(x) &\le \mathbb{E}\left\{|\nabla^t f|_t(X_t)\exp\left[-\int_s^t K(u, X_u)\mathrm{d}u\right]\middle|X_s = x\right\}\\ &\le \mathbb{E}\left\{|\nabla^t f|_t^p(X_t)\exp\left[-p\int_s^t K(u, X_u)\mathrm{d}u\right]\middle|X_s = x\right\}^{1/p}.\end{aligned}$$

thus, (2) holds.

To prove (3) and (4), let $p \in (1, 2]$. By an approximation argument, we assume that $f \in C^\infty(M)$ and is constant outside a compact set such that $\|L_t f\|_\infty$ is locally bounded for any $p > 1$. Without loss generality, we only prove for $s = 0$. In this case, by Kolmogorov equations, Theorem 2.3 and using (2) for $p = 1$, we obtain at point x that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}u}P_{0,u}(P_{u,t}f^{2/p})^p(x) = P_{0,u}\left\{p(p-1)(P_{u,t}f^{2/p})^{p-2}|\nabla^{(u)}P_{u,t}f^{2/p}|_u^2\right\}\\ &\le p(p-1)\mathbb{E}^x\left\{(P_{u,t}f^{2/p})^{p-2}(X_u)\mathbb{E}\left[|\nabla^t f^{2/p}|_t(X_t)e^{-p\int_u^t K(r,X_r)\mathrm{d}r}\middle|\mathscr{F}_u\right]\right\}\\ &\le \frac{4(p-1)}{p}\mathbb{E}^x\left\{(P_{u,t}f^{2/p})^{p-2}(X_u)(P_{u,t}f^{\frac{2(2-p)}{p}})(X_u)\mathbb{E}\left[|\nabla^t f|_t^2(X_t)e^{-2\int_u^t K(r,X_r)\mathrm{d}r}\middle|\mathscr{F}_u\right]\right\}\end{aligned}$$

Since $2-p \in [0,1]$, by the Jensen inequality

$$P_{u,t} f^{\frac{2(2-p)}{p}} \le (P_{u,t} f^{2/p})^{2-p}.$$

Combining this with the Markov property, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}u} P_{0,u}(P_{u,t} f^{2/p})^p(x) \le \frac{4(p-1)}{p} \mathbb{E}^x \left\{ |\nabla^t f|_t^2(X_t) e^{-2\int_u^t K(r,X_r)\mathrm{d}r} \right\}, \quad u \in [0,t].$$

This implies (3) for $s=0$ by taking integral over $[0,t]$. Similarly,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}u} P_{0,u}(P_{u,t} f)^p &= p(p-1) P_{0,u}\{(P_{u,t} f)^{p-2} |\nabla^{(u)} P_{u,t} f|_u^2\} \\
&\ge \frac{p(p-1)\left(\mathbb{E}^x |\nabla^{(u)} P_{u,t} f|_u(X_u) e^{-\int_0^u K(r,X_r)\mathrm{d}r}\right)^2}{\mathbb{E}^x \left\{ (P_{u,t} f)^{2-p}(X_u) e^{-2\int_0^u K(r,X_r)\mathrm{d}r} \right\}} \\
&\ge \frac{p(p-1)|\nabla^0 P_{0,t} f|_0^2}{\mathbb{E}^x \left\{ (P_{u,t} f)^{2-p}(X_u) e^{-2\int_0^u K(r,X_r)\mathrm{d}r} \right\}}.
\end{aligned}$$

Integrating over $[0,t]$, we prove (4) for $s=0$. □

When the metric is independent of t and K is constant, the above equivalences are well-known (see e.g.[5, 4, 6]). For more general case of $K \in C(M)$, we refer the readers to [34, Theorem 2.3.1].

4.2 Coupling for the L_t-diffusion process

Next, we aim to present equivalent Harnack inequalities and transportation-cost inequalities for the curvature $\mathcal{R}_t^Z$ low bound. To this end, let us first introduce two crucial couplings for the diffusion process generated by L_t, namely, the couplings by parallel and reflecting displacement. When the metric is independent of t, the reflecting coupling on manifold was first introduced by Kendall (see [16]) and further refined by Cranston [11] for constant metric case. For the time-inhomogeneous case, recently, Kuwada [19, 18] construct these couplings by approximation via geodesic random walks. Here, we will adopt the method inspired from [29, Theorem2.1.1 and Proposition 2.5.1], where these couplings were constructed by solving SDEs on $M \times M$ with singular coefficients on the cut-locus for the case with constant metric. Compared with the constructions [19], our argument is more straightforward.

Recall that $\mathrm{Cut}_t(x)$ is the set of the g_t-cut-locus of x on M. Then, the g_t-cut-locus Cut_t and the space time cutlocus $\mathrm{Cut}_{\mathrm{ST}}$ are defined by

$$\begin{aligned}
\mathrm{Cut}_t &= \{(x,y) \in M \times M | y \in \mathrm{Cut}_t(x)\}; \\
\mathrm{Cut}_{\mathrm{ST}} &= \{(t,x,y) \in [0,T_c) \times M \times M | (x,y) \in \mathrm{Cut}_t\}.
\end{aligned}$$

Set $D(M) := \{(x,x) | x \in M\}$. For a smooth curve γ and smooth vector fields $U,\ V$ along γ, the index form $I_\gamma^t(U,V)$ is given by

$$I_\gamma^t(U,V) = \int_\gamma \left(\left\langle \nabla_{\dot\gamma}^t U, \nabla_{\dot\gamma}^t V \right\rangle_t - \langle R_t(U,\dot\gamma)\dot\gamma, V\rangle_t \right)(\gamma(s))\mathrm{d}s,$$

where R_t is the Ricci tensor with respect to g_t.

For any $(x,y) \notin \mathrm{Cut}_t$ with $x \neq y$, let $\{J_i^t\}_{i=1}^{d-1}$ be Jacobi fields along the minimal geodesic γ from x to y with respect to g_t such that at x and y, $\{J_i^t, \dot\gamma : 1 \le i \le d-1\}$ is an orthonormal basis. Let

$$I_t^Z(x,y) := \sum_{i=1}^{d-1} I_t^\gamma(J_i^t, J_i^t) + Z_t\rho_t(\cdot,y)(x) + Z_t\rho_t(x,\cdot)(y).$$

Moreover, let $P_{x,y}^t : T_xM \to T_yM$ be the parallel transform along the geodesic γ, and let

$$M_{x,y}^t : T_xM \to T_yM;\ v \mapsto P_{x,y}^t v - 2\langle v, \dot\gamma\rangle_t(x)\dot\gamma(y)$$

be the mirror reflection. Then $P_{x,y}^t$ and $M_{x,y}^t$ are smooth outside Cut_t and $D(M)$. For convenience, we set $P_{x,x}^t$ and $M_{x,x}^t$ be the identity for $x \in M$.

Theorem 4.2. *Let $x \neq y$ and $0 < T < T_c$ be fixed. Let $U : [0,T]\times M\times M \to TM^2$ be C^1-smooth in $(\mathrm{Cut}_{\mathrm{ST}} \cup [0,T]\times D(M))^c$.*

(1) *There exist two Brownian motion B_t and $\tilde B_t$ on the probability space $(\Omega, \{\mathscr{F}_t\}_{t\ge0}, \mathbb{P})$ such that*

$$\mathbf{1}_{\{(X_t,\tilde X_t)\notin \mathrm{Cut}_t\}}\mathrm{d}\tilde B_t = \mathbf{1}_{\{(X_t,\tilde X_t)\notin \mathrm{Cut}_t\}}\tilde u_t^{-1}P_{X_t,\tilde X_t}^t u_t \mathrm{d}B_t$$

holds, where X_t with lift u_t and $\tilde X_t$ with lift $\tilde u_t$ solve the equation

$$\begin{cases} \mathrm{d}X_t = \sqrt{2}u_t \circ \mathrm{d}B_t + Z_t(X_t)\mathrm{d}t, & X_0 = x,\\ \mathrm{d}\tilde X_t = \sqrt{2}\tilde u_t \circ \mathrm{d}\tilde B_t + \left\{Z_t(\tilde X_t) + U(t,X_t,\tilde X_t)\mathbf{1}_{\{X_t\neq\tilde X_t\}}\right\}\mathrm{d}t, & \tilde X_0 = y. \end{cases} \tag{4.2}$$

Moreover,

$$\begin{aligned}\mathrm{d}\rho_t(X_t,\tilde X_t) \le \bigg\{&\frac12\int_\gamma \partial_t g_t(\dot\gamma(s),\dot\gamma(s))\mathrm{d}s + I_t^Z(X_t,\tilde X_t)\\ &+ \left\langle U(t,X_t,\tilde X_t), \nabla^t\rho_t(X_t,\cdot)(\tilde X_t)\right\rangle_t \mathbf{1}_{\{X_t\neq\tilde X_t\}}\bigg\}\mathrm{d}t. \end{aligned}\tag{4.3}$$

(2) *The first assertion in (1) holds with $M_{X_t,\tilde X_t}^t$ in place of $P_{X_t,\tilde X_t}^t$. In this case,*

$$\begin{aligned}\mathrm{d}\rho_t(X_t,\tilde X_t) \le 2\sqrt{2}\mathrm{d}b_t + \bigg\{&\frac12\int_\gamma \partial_t g_t(\dot\gamma(s),\dot\gamma(s))\mathrm{d}s + I_t^Z(X_t,\tilde X_t)\\ &+ \left\langle U(t,X_t,\tilde X_t), \nabla^t\rho_t(X_t,\cdot)(\tilde X_t)\right\rangle_t \mathbf{1}_{\{X_t\neq\tilde X_t\}}\bigg\}\mathrm{d}t \end{aligned}\tag{4.4}$$

holds for some one-dimensional Brownian motion b_t.

Proof. Without loss generality, we only deal with the reflecting coupling case for $U = 0$. Moreover, to save space, we only prove (2).

(a) Construction of couplings. Recall that u_t, the horizontal lift of X_t, satisfies the following SDE

$$\begin{cases} \mathrm{d}u_t = \sqrt{2}\sum_{i=1}^d H_i^t(u_t)\circ \mathrm{d}B_t^i + H_{Z_t}^t(u_t)\mathrm{d}t - \frac12\sum_{\alpha,\beta}\partial_t g_t(u_te_\alpha, u_te_\beta)V_{\alpha\beta}(u_t)\mathrm{d}t,\\ u_0 \in \mathcal{O}_t(M),\ \mathbf{p}(u_0) = x. \end{cases}$$

For given $x \neq y$ with $(x,y) \notin \mathrm{Cut}_t$, let γ be the minimal geodesic from x to y.

Following the line of [28], we approximate $M^t_{x,y}$ by smooth operators vanishing in a neighborhood of these sets. More precise, for any $n \geq 1$ and $\varepsilon \in (0,1)$, let $h_{n,\varepsilon} \in C^\infty(\mathbb{R}^+)$ such that

$$0 \leq h_{n,\varepsilon} \leq (1-\varepsilon),\ h_{n,\varepsilon}|_{[0,\frac{1}{2n}]} = 0,\ h_{n,\varepsilon}|_{[\frac{1}{n},\infty)} = 1-\varepsilon.$$

Next, let $g_n \in C^\infty$, such that $0 \leq g_n \leq 1$, $g_n|_{[0,\frac{1}{2n}]} = 0$, $g_n|_{[\frac{1}{n},\infty)} = 1$. Now define

$$\overline{h}_{n,\varepsilon}(t,x,y) = h_{n,\varepsilon}(\rho_{g_t\otimes g_t}((x,y),\mathrm{Cut}_t)),\ \overline{g}_n(t,x,y) = g_n(\rho_t(x,y)),$$

where $\rho_{g_t\otimes g_t}$ is the Riemannian distance on $M \times M$. Let $\tilde{u}^{n,\varepsilon}_t$ and $\tilde{X}^{n,\varepsilon}_t := \mathbf{p}\tilde{u}^{n,\varepsilon}_t$ solve the SDE

$$\begin{cases} \mathrm{d}\tilde{u}^{n,\varepsilon}_t = \sqrt{2}(\overline{h}_{n,\varepsilon}\overline{g}_n)(t,X_t,\tilde{X}^{n,\varepsilon}_t)\displaystyle\sum_{i=1}^d H^t_i(\tilde{u}^{n,\varepsilon}_t)\circ \mathrm{d}\tilde{B}^i_t - \frac{1}{2}\sum_{\alpha,\beta}\partial_t g_t(\tilde{u}^{n,\varepsilon}_t e_\alpha, \tilde{u}^{n,\varepsilon}_t e_\beta)V_{\alpha\beta}(\tilde{u}^{n,\varepsilon}_t)\mathrm{d}t \\ \qquad + \sqrt{2(1-(\overline{h}_{n,\varepsilon}\overline{g}_n)^2(t,X_t,\tilde{X}^{n,\varepsilon}_t))}\displaystyle\sum_{i=1}^d H^t_i(\tilde{u}^{n,\varepsilon}_t)\circ \mathrm{d}B'^i_t + H^t_{Z_t}(\tilde{u}^{n,\varepsilon}_t)\mathrm{d}t, \\ \tilde{u}^{n,\varepsilon}_0 \in \mathcal{O}_t(M),\ \mathbf{p}(\tilde{u}^{n,\varepsilon}_0) = y, \end{cases} \tag{4.5}$$

where B'_t is a Brownian motion on $\mathbb{R}^d$ independent of B_t, and $\mathrm{d}\tilde{B}_t = (\tilde{u}^{n,\varepsilon}_t)^{-1}M^t_{X_t,\tilde{X}^{n,\varepsilon}_t}u_t\mathrm{d}B_t$. Since the coefficients involved in (4.5) are at least $C^{1,1}$, the solution $\tilde{u}^{n,\varepsilon}_t$ exists uniquely.

Let us observe that $(u_t, \tilde{u}^{n,\varepsilon}_t)$ is generated by

$$\begin{aligned} L^{n,\varepsilon}_{\mathcal{O}_t(M)}(t)(u_t,\tilde{u}^{n,\varepsilon}_t) :=& \Delta_{\mathcal{O}_t(M)}(u_t) + \Delta_{\mathcal{O}_t(M)}(\tilde{u}^{n,\varepsilon}_t) + H^t_{Z_t}(u_t) + H^t_{Z_t}(\tilde{u}^{n,\varepsilon}_t) \\ &+ \overline{h}_{n,\varepsilon}\overline{g}_n(t,X_t,\tilde{X}^{n,\varepsilon}_t)\sum_{i,j=1}^d \left\langle M^t_{X_t,\tilde{X}^{n,\varepsilon}_t}u_t e_i, \tilde{u}^{n,\varepsilon}_t e_j\right\rangle_t H^t_{u_t e_i}H^t_{\tilde{u}^{n,\varepsilon}_t e_j} \\ &- \frac{1}{2}\sum_{\alpha,\beta}[\partial_t g_t(u_t e_\alpha, u_t e_\beta)V_{\alpha,\beta}(u_t) + \partial_t g_t(\tilde{u}^{n,\varepsilon}_t e_\alpha, \tilde{u}^{n,\varepsilon}_t e_\beta)V_{\alpha,\beta}(u^{n,\varepsilon}_t)]\mathrm{d}t, \end{aligned}$$

Next, let

$$L^{n,\varepsilon}_M(t)(x,y) := \Delta_t(x) + \Delta_t(y) + Z_t(x) + Z_t(y) + \overline{h}_{n,\varepsilon}\overline{g}_n(t,x,y)\sum_{i,j=1}^d \left\langle M^t_{x,y}X_i, Y_j\right\rangle_t X_i Y_j,$$

where $\{X_i\}$ and $\{Y_i\}$ are orthonormal bases at x and y respectively. It is easy to see that $(X_t, \tilde{X}^{n,\varepsilon}_t) := (\mathbf{p}u_t, \mathbf{p}\tilde{u}^{n,\varepsilon}_t)$ is generated by $L^{n,\varepsilon}_M(t)$ and hence, is a coupling of the L_t-diffusion processes as the marginal operators of $L^{n,\varepsilon}_M(t)$ coincide with L_t.

Since in some neighborhood of $\mathrm{Cut}_{ST}\cup[0,T]\times D(M)$ the coupling is independent and hence, behaves as a g_t-Brownian motion with drift on $M \times M$. Thus following from [17, Theorem2], the Itô formula for radial process $\rho_t(o,X_t)$, one has

$$\begin{aligned} \mathrm{d}\rho_t(X_t,\tilde{X}^{n,\varepsilon}_t) =& 2\sqrt{(\overline{h}_{n,\varepsilon}\overline{g}_n)(t,X_t,\tilde{X}^{n,\varepsilon}) + 1}\ \mathrm{d}b^{n,\varepsilon}_t - \mathrm{d}l^{n,\varepsilon}_t + \mathrm{d}I^{n,\varepsilon}_t + \partial_t\rho_t(X_t,\tilde{X}^{n,\varepsilon}_t)\mathrm{d}t \\ &+ \mathbf{1}_{(\mathrm{Cut_{ST}}\cup[0,T]\times D(M))^c}[\overline{h}_{n,\varepsilon}\overline{g}_n I^Z + (1-\overline{h}_{n,\varepsilon}\overline{g}_n)S](t,X_t,\tilde{X}^{n,\varepsilon}_t)\mathrm{d}t, \end{aligned} \tag{4.6}$$

where $b^{n,\varepsilon}_t$ is an one-dimensional Brownian motion, $l^{n,\varepsilon}_t$ is an increasing process which increases only when $(X_t, \tilde{X}^{n,\varepsilon}_t) \in \mathrm{Cut}_t$, $I^{n,\varepsilon}_t$ is the local time at $D(M)$ (Note that when $d \geq 2$ a strictly

elliptic diffusion process on $M \times M$ never visit $D(M)$ so that $I_t^{n,\varepsilon} = 0$), and

$$S(t, x, y) := L_t\rho_t(\cdot, y)(x) + L_t\rho_t(x, \cdot)(y);$$
$$I^Z(t, x, y) := I_t^Z(x, y).$$

Now, let $\mathbb{P}_{n,\varepsilon}^{(x,y)}$ be the distribution of $(X_t, \tilde{X}_t^{n,\varepsilon})$, which is a probability measure on the path space $M_x^T \times M_y^T$, where

$$M_x^T := \{\gamma \in C([0, T], M) : \gamma_0 = x\}$$

is equipped with the σ-field $\mathscr{F}_x^T$ induced by all measurable cylindric functions. Since $(M_x^T, \mathscr{F}_x^T)$ is metrizable with a Polish metric $\widetilde{\rho}$, $M_x^T \times M_y^T$ is metrizable as a Polish space too. Since $\{\mathbb{P}_{n,\varepsilon}^{x,y} : n \geq 1, \varepsilon > 0\}$ is a family of couplings for $\mathbb{P}^x$ and $\mathbb{P}^y$, it is tight by [23, Lemma 4]. Therefore, for each $\varepsilon > 0$, there exists a probability measure $\mathbb{P}_\varepsilon^{x,y}$ and sub sequence $\{n_k\}$ such that $\mathbb{P}_{n_k,\varepsilon}^{x,y} \to \mathbb{P}_\varepsilon^{x,y} (k \to \infty)$ weakly and hence $\mathbb{P}_\varepsilon^{x,y}$ is once again a coupling of $\mathbb{P}^x$ and $\mathbb{P}^y$. Moreover, let $\varepsilon_l \to 0$ so that $\mathbb{P}_{\varepsilon_l}^{x,y} \to \mathbb{P}^{x,y}$ weakly, then $\mathbb{P}^{x,y}$ is also a coupling of $\mathbb{P}^x$ and $\mathbb{P}^y$.

Let $(X_t, \tilde{X}_t)$ be the coordinate (or càdlàg) process in $(M_x^T \times M_y^T, \mathscr{F}_x^T \times \mathscr{F}_y^T)$ and let $\{\mathscr{F}_t\}_{t \geq 0}$ be the natural filtration. Similar to that explained in the proof of [3, Theorem 2], we first define

$$\tilde{L}_t(x, y) = L_t(x) + L_t(y) + 1_{(\mathrm{Cut}_t \cup D(M))^c}(x, y) \sum_{i,j=1}^{d} \left\langle M_{x,y}^t X_i, Y_j \right\rangle_t X_i Y_j.$$

It is trivial to see that $\mathbb{P}^{x,y}$ solves the martingale problem for $\tilde{L}_t$ up to the coupling time, i.e. for any $f \in C_0^\infty(M \times M/D(M))$,

$$f(X_t, \tilde{X}_t) - \int_0^t \tilde{L}_s f(X_s, \tilde{X}_s)\mathrm{d}s$$

is a $\mathbb{P}^{x,y}$-martingale w.r.t $\mathscr{F}$ up to $\inf\{t \in [0, T] : X_t = \tilde{X}_t\}$. Then $(X_t, \tilde{X}_t)$ under $\mathbb{P}^{x,y}$ is a coupling of the L_t-diffusion process starting from (x, y), is a weak solution and the solution of (4.2).

(*b*) Proof of (4.4). We only consider noncompact M, for the compact case the proof is simpler by dropping the stopping time τ below. Let $\mathbf{B}$ be a fixed bounded smooth open domain in M. Given $N \geq 1$, the Laplacian comparison theorem implies that there exists a constant $C > 0$ such that $S(t, x, y) \leq C$ for all $(t, x, y) \in [0, T] \times \mathbf{B} \times \mathbf{B}$ with $\rho_t(x, y) \geq \frac{1}{N}$. We first claim that

$$\{t \in [0, T] \mid (X_t, \tilde{X}_t^{n,\varepsilon}) \in \mathrm{Cut}_t\} \text{ and } \{t \in [0, T] \mid (X_t, \tilde{X}_t^{\varepsilon}) \in \mathrm{Cut}_t\} \tag{4.7}$$

have Lebesgue measure zero almost surely. This assertion can be checked similar as [29, Lemma 2.1.2] by observing that $L_M^{n,\varepsilon}(t)$ is strictly elliptic, then $\mathbb{P}^{n,\varepsilon}(A) := \mathbb{P}((X_t, \tilde{X}_t^{n,\varepsilon}) \in A)$ has a density $p_t^{n,\varepsilon}(x, y)$ with respect to the product volume measure $g_t \otimes g_t$. Since by (4.7), $\mathbf{1}_{\mathrm{Cut}_t}(X_t, \tilde{X}_t^{n,\varepsilon}) = 0$ a.s., it follows from (4.6) that, when $(t, x, y) \in [0, T] \times \mathbf{B} \times \mathbf{B}$ with $\rho_t(x, y) \geq \frac{1}{N}$,

$$\begin{aligned} \mathrm{d}\rho_t(X_t, \tilde{X}_t^{n,\varepsilon}) =& 2\sqrt{(\overline{h}_{n,\varepsilon}\overline{g}_n)(t, X_t, \tilde{X}^{n,\varepsilon}) + 1}\ \mathrm{d}b_t^{n,\varepsilon} - \mathrm{d}\tilde{l}_t^{n,\varepsilon} + \partial_t\rho_t(X_t, \tilde{X}_t^{n,\varepsilon})\mathrm{d}t \\ &+ [\overline{h}_{n,\varepsilon}\overline{g}_n J + (1 - \overline{h}_{n,\varepsilon}\overline{g}_n)C](t, X_t, \tilde{X}_t^{n,\varepsilon})\mathrm{d}t, \end{aligned} \tag{4.8}$$

where $J \in C([0,T] \times M \times M)$, $J \geq I^Z$ on $(\mathrm{Cut}_{ST} \cup [0,T] \times D(M))^c$ and $\tilde{l}_t^{n,\varepsilon}$ is a larger increasing process. Now let $f \in C^2(\mathbb{R})$ with $f' \geq 0$ and $f'|_{[0,1/N]} = 0$. By the Itô's formula we obtain from (4.8) that, for any $n \geq N$ (recall that $\overline{g}_n(t,x,y) = 1$ if $\rho_t(x,y) \geq 1/n$),

$$
\begin{aligned}
& f \circ \rho_t(X_{t\wedge\tau_{n,\varepsilon}}, X^{n,\varepsilon}_{t\wedge\tau_{n,\varepsilon}}) \\
& - \int_0^{t\wedge\tau_{n,\varepsilon}} \left\{2(\overline{h}_{n,\varepsilon}+1)f'' \circ \rho + (\partial_s \rho + \overline{h}_{n,\varepsilon} J + (1-\overline{h}_{n,\varepsilon})C) f' \circ \rho\right\}(s, X_s, \tilde{X}_s^{n,\varepsilon}) \mathrm{d}s
\end{aligned}
$$

is a supermartingale, where $\tau_{n,\varepsilon} := \inf\{t \geq 0 : (X_t, \tilde{X}_t^{n,\varepsilon}) \notin \mathbf{B} \times \mathbf{B}\}$. Here and what follows, $\rho(t,x,y) := \rho_t(x,y)$. Therefore, for the coordinate process (ξ_t, η_t) with $\tau := \inf\{t \geq 0 : (\xi_t, \eta_t) \notin \mathbf{B} \times \mathbf{B}\}$,

$$
\begin{aligned}
S_t^{n,\varepsilon}(f) :=& f \circ \rho_t(\xi_{t\wedge\tau}, \eta_{t\wedge\tau}) \\
& - \int_0^{t\wedge\tau} \left\{2(\overline{h}_{n,\varepsilon}+1)f'' \circ \rho + \left(\partial_s \rho + \overline{h}_{n,\varepsilon} J + (1-\overline{h}_{n,\varepsilon})C\right) f' \circ \rho\right\}(s, \xi_s, \eta_s) \mathrm{d}s
\end{aligned}
$$

is a $\mathbb{P}^{x,y}_{n,\varepsilon}$-supermartingale. Thus, for any $t > t'$ and $\mathscr{F}_{t'}$-measurable nonnegative $g \in C_b(M_x^T \times M_y^T)$, one has

$$
\mathbb{E}^{x,y}_{n,\varepsilon} g S_t^{n,\varepsilon}(f) \leq \mathbb{E}^{x,y}_{n,\varepsilon} g S_{t'}^{n,\varepsilon}(f), \quad n \geq N, \tag{4.9}
$$

where and similarly in the sequel for $\mathbb{E}^{x,y}_{\varepsilon}$ and $\mathbb{E}^{x,y}$ with respect to $\mathbb{P}^{x,y}_{\varepsilon}$ and $\mathbb{P}^{x,y}$, $\mathbb{E}^{x,y}_{n,\varepsilon}$ is the expectation with respect to $\mathbb{P}^{x,y}_{n,\varepsilon}$.

Since $\{t \in [0,T] \mid (X_t, \tilde{X}_t^{\varepsilon}) \in \mathrm{Cut}_t\}$ is a null-set w.r.t. the Lebesgue measure, one has

$$
\mathbb{E}^{x,y}_{\varepsilon} \left(\int_0^T \mathbf{1}_{\mathrm{Cut}_t}(\xi_t, \eta_t) \mathrm{d}t \right) = 0.
$$

So for any $\delta > 0$ there exists $m \geq 1$ such that

$$
\int_0^t \mathbb{P}^{x,y}_{\varepsilon}((\xi_s, \eta_s) \in C_m^s) \mathrm{d}s \leq \delta, \tag{4.10}
$$

where $C_m^s := \{(x,y) : \rho_{g_s \otimes g_s}((x,y), \mathrm{Cut}_s) \leq \frac{1}{m}\}$. Since C_m^s is closed, we have

$$
\overline{\lim_{k\to\infty}} \mathbb{P}^{x,y}_{n_k,\varepsilon}((\xi_s, \eta_s) \in C_m^s) \leq \mathbb{P}^{x,y}_{\varepsilon}((\xi_s, \eta_s) \in C_m), \quad s \geq 0.
$$

Hence,

$$
\overline{\lim_{k\to\infty}} \int_0^t \mathbb{P}^{x,y}_{n_k,\varepsilon}((\xi_s, \eta_s) \in C_m^s) \mathrm{d}s \leq \delta. \tag{4.11}
$$

By (4.9), (4.10), (4.11) and the continuity of the path, note that $\overline{h}_n = 1 - \varepsilon$ on C_m^t for $n \geq m$,

we have for some constant $C_1 > 0$,

$$
\begin{aligned}
\mathbb{E}_\varepsilon^{x,y} S_t^\varepsilon(f) g =& \mathbb{E}_\varepsilon^{x,y} g\Big\{ f \circ \rho_t(\xi_{t\wedge\tau}, \eta_{t\wedge\tau}) - \int_0^t \mathbf{1}_{\{s<\tau\}} \big[2(2-\varepsilon) f'' \circ \rho \\
& + (\partial_s \rho + (1-\varepsilon) J + \varepsilon C) f' \circ \rho\big](s, \xi_s, \eta_s) \mathrm{d}s \Big\} \\
=& \lim_{k\to\infty} \mathbb{E}_{n_k,\varepsilon}^{x,y} g\Big\{ f \circ \rho_t(\xi_{t\wedge\tau}, \eta_{t\wedge\tau}) - \int_0^t \mathbf{1}_{\{s<\tau\}} \big[2(2-\varepsilon) f'' \circ \rho \\
& + (\partial_s \rho + (1-\varepsilon) J + \varepsilon C) f' \circ \rho\big](s, \xi_s, \eta_s) \mathrm{d}s \Big\} \\
\le& \varliminf_{k\to\infty} \mathbb{E}_{n_k,\varepsilon}^{x,y} S_t^{n,\varepsilon}(f) g + \delta C_1 \le \varliminf_{k\to\infty} \mathbb{E}_{n_k,\varepsilon}^{x,y} g S_{t'}^{n_k,\varepsilon}(f) + \delta C_1 \\
\le& \varliminf_{k\to\infty} \mathbb{E}_{n_k,\varepsilon}^{x,y} g\Big\{ f \circ \rho_t(\xi_{t'\wedge\tau}, \eta_{t'\wedge\tau}) - \int_0^{t'} \mathbf{1}_{\{s<\tau\}} \big[2(2-\varepsilon) f'' \circ \rho \\
& + ((1-\varepsilon) J + \varepsilon C + \partial_s \rho) f' \circ \rho\big](s, \xi_s, \eta_s) \mathrm{d}s \Big\} + 2\delta C_1 \\
=& \mathbb{E}_\varepsilon^{x,y} g S_{t'}^\varepsilon(f) + 2\delta C_1.
\end{aligned}
$$

Letting $\delta \to 0$, we obtain

$$
\mathbb{E}_\varepsilon^{x,y} g S_t^\varepsilon(f) \le \mathbb{E}_\varepsilon^{x,y} g S_{t'}^\varepsilon(f). \tag{4.12}
$$

Let

$$
S_t(f) := f \circ \rho_{t\wedge\tau}(\xi_{t\wedge\tau}, \eta_{t\wedge\tau}) - \int_0^{t\wedge\tau} \left[(J + \partial_s \rho) f' \circ \rho + 4 f'' \circ \rho \right] (s, \xi_s, \eta_s) \mathrm{d}s.
$$

Then $S_t^\varepsilon(f) \to S_t(f)$ uniformly as $\varepsilon \to 0$. By the continuity of the path and (4.12) we obtain

$$
\begin{aligned}
\mathbb{E}^{x,y} g S_t(f) &= \lim_{l\to\infty} \mathbb{E}_{\varepsilon_l}^{x,y} g\Big\{ f \circ \rho_{t\wedge\tau}(\xi_{t\wedge\tau}, \eta_{t\wedge\tau}) \\
&\quad - \int_0^t \mathbf{1}_{\{s<\tau\}} \left[(J + \partial_s \rho) f' \circ \rho + 4 f'' \circ \rho \right] (s, \xi_s, \eta_s) \mathrm{d}s \Big\} \\
&= \lim_{l\to\infty} \mathbb{E}_{\varepsilon_l}^{x,y} g S_t^{\varepsilon_l}(f) \le \lim_{l\to\infty} \mathbb{E}_{\varepsilon_l}^{x,y} g S_{t'}^{\varepsilon_l}(f) = \mathbb{E}^{x,y} g S_{t'}(f).
\end{aligned}
$$

This means that $S_t(f)$ is a $\mathbb{P}^{x,y}$-supermartingale as $t > t'$ and $\mathscr{F}_{t'}$-measurable nonnegative $g \in C_b(M_x^T \times M_y^T)$ are arbitrary.

Now, let $f \in C^2(\mathbb{R})$ with $f' \ge 0$ be fixed. For any $N \ge 1$, let

$$
T_N := \inf\{t \ge 0 : \rho_t(\xi_t, \eta_t) \le 1/N\}.
$$

One has $T_N \to T$ as $N \to \infty$. Let us take $\tilde{f} \in C^2(\mathbb{R})$ such that $\tilde{f}' \ge 0$, $\tilde{f}'|_{[0,1/(2N)]} = 0$ and $\tilde{f} = f$ on $[1/N, \infty)$. Let

$$
\mathrm{d}N_t(f) := \mathrm{d}f \circ \rho_t(\xi_t, \eta_t) - \left[(J + \partial_t \rho) f' \circ \rho + 4 f'' \circ \rho \right] (t, \xi_t, \eta_t) \mathrm{d}t,\ N_0(f) := f \circ \rho_0(x, y). \tag{4.13}
$$

Then due to the concrete choice of $\tilde{f}$, one has $N_{t\wedge T_N\wedge\tau}(f) = S_{t\wedge T_N\wedge\tau}(\tilde{f})$ and hence is a $\mathbb{P}^{x,y}$-supermartingale. Letting $N \to \infty$, we conclude that $N_{t\wedge T\wedge\tau}(f)$ is a $\mathbb{P}^{x,y}$-supermartingale too. In particular, for $f(r) := r$,

$$
S_t := \rho_{t\wedge T\wedge\tau}(\xi_{t\wedge T\wedge\tau}, \eta_{t\wedge T\wedge\tau}) - \int_0^{t\wedge T\wedge\tau} (J + \partial_s \rho)(s, \xi_s, \eta_s) \mathrm{d}s
$$

is a bounded continuous $\mathbb{P}^{x,y}$-supermartingale. By Doob-Meyer's decomposition and the non-explosion (i.e. $\tau \to \infty$ as $\mathbf{B} \to M$), one has

$$\mathrm{d}\rho_t(\xi_t, \eta_t) = \mathrm{d}M_t + (J + \partial_t \rho)(t, \xi_t, \eta_t)\mathrm{d}t - \mathrm{d}\tilde{l}_t, \quad t < T,$$

where M_t is a local martingale and $\tilde{l}_t$ is a predictable increasing process.

(*c*) Choose $f(r) = e^{Nr}$ in (4.13). Letting N goes to infinity, we obtain $\mathrm{d}\langle M_t, M_t\rangle = 8\mathrm{d}t$. As long as $(t, X_t, \tilde{X}_t)$ stays away from $[0,T] \times D(M)$ and $\mathrm{Cut}_{\mathrm{ST}}$, by (2.4) and the Itô formula, one has

$$\mathrm{d}\rho_t(X_t, \tilde{X}_t) = 2\sqrt{2}\mathrm{d}b_t + \left[\frac{1}{2}\int_\gamma \partial_t g_t(\dot{\gamma}(s), \dot{\gamma}(s))\mathrm{d}s + I_t^Z(X_t, \tilde{X}_t)\right]\mathrm{d}t.$$

Therefore, when J is chosen as a modification of I^Z on $[0,T] \times D(M)$ and $\mathrm{Cut}_{\mathrm{ST}}$, l_t is an increasing process supporting only on $\{t : (X_t, \tilde{X}_t) \in \mathrm{Cut}_t\}$.

□

As a consequence of Theorem 4.2, we have the following alternative proof of "(1) implying (2)" in Theorem 4.1 for $p = 1$. See also for g_t being the Ricci flow in [22].

Corollary 4.3. *Assume that $\mathcal{R}_t^Z \geq K(t)$ for some $K \in C^1([0, T_c))$. Then*

$$|\nabla^s P_{s,t} f|_s \leq e^{-\int_s^t K(r)\mathrm{d}r} P_{s,t}|\nabla^t f|_t, \quad f \in C_b^1(M),\ 0 \leq s \leq t < T_c.$$

Proof. To apply Theorem 4.2, we first observe that

$$\frac{1}{2}\int_0^{\rho_t} \partial_t g_t(\dot{\gamma}(s), \dot{\gamma}(s))\mathrm{d}s + I_t^Z(x,y) \leq \int_0^{\rho_t} \mathcal{R}_t^Z(\dot{\gamma}(s), \dot{\gamma}(s))\mathrm{d}s,\ x, y \in M.$$

where $\gamma : [0, \rho_t(x,y)] \to M$ is the minimal geodesic from x to y.

Now, let $U = 0$ and $(X_t, \tilde{X}_t)$ be the coupling by parallel displacement for $X_0 = x,\ \tilde{X}_0 = y$. By Theorem 4.2 for $U = 0$, we have

$$\mathrm{d}\rho_t(X_t, \tilde{X}_t) \leq -K(t)\rho_t(X_t, \tilde{X}_t)\mathrm{d}t.$$

Thus, $\rho_t(X_t, \tilde{X}_t) \leq e^{-\int_s^t K(r)\mathrm{d}r}\rho_s(x,y)$. So, by the dominated convergence theorem,

$$\begin{aligned}
|\nabla^s P_{s,t} f(x)|_s &\leq \limsup_{y \to x} \frac{\mathbb{E}\left\{|f(X_t) - f(\tilde{X}_t)| \mid (X_s, \tilde{X}_s) = (x,y)\right\}}{\rho_s(x,y)} \\
&\leq e^{-\int_s^t K(r)\mathrm{d}r} \limsup_{y \to x} \mathbb{E}\left\{\frac{|f(X_t) - f(\tilde{X}_t)|}{\rho_t(X_t, \tilde{X}_t)} \;\middle|\; (X_s, \tilde{X}_s) = (x,y)\right\} \\
&= e^{-\int_s^t K(r)\mathrm{d}r} P_{s,t}|\nabla^t f|_t.
\end{aligned}$$

□

4.3 Some other equivalent inequalities for (4.1)

In this section, we want to give virous equivalent statements for the new curvature condition by the derivative formula and coupling method.

Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing function, we define a cost function

$$C_t(x, y) = \varphi(\rho_t(x, y)).$$

To the cost function C_t, we associate the Monge-Kantorovich minimization between two probability measures on M,

$$W_{C_t}(\mu, \nu) = \inf_{\eta \in \mathscr{C}(\mu,\nu)} \int_{M\times M} C_t(x, y) \mathrm{d}\eta(x, y), \tag{4.14}$$

where $\mathscr{C}(\mu, \nu)$ is the set of all probability measures on $M \times M$ with marginal μ and ν. We denote

$$W_{p,t}(\mu, \nu) = (W_{\rho_t^p}(\nu, \mu))^{1/p}$$

the Wasserstein distance associated to $p \geq 1$.

Our main task in this subsection is to prove the following result.

Theorem 4.4. *Let $p \geq 1$, and $p_{s,t}(x, y)$ be the heat kernel of $P_{s,t}$ w.r.t. measure μ_t equivalent to the volume measure w.r.t. g_t. Then the following assertions are equivalent to each other.*

(1) (4.1) *holds for $K \in C([0, T_c))$.*

(2) *For any $x, y \in M$ and $0 \leq s \leq t < T_c$,*

$$W_{p,t}(\delta_x P_{s,t}, \delta_y P_{s,t}) \leq \rho_s(x, y) e^{-\int_s^t K(r)\mathrm{d}r}.$$

(2′) *For any $\nu_1, \nu_2 \in \mathscr{P}(M)$, the space of all the probability measure on M, and $0 \leq s \leq t < T_c$,*

$$W_{p,t}(\nu_1 P_{s,t}, \nu_2 P_{s,t}) \leq W_{p,s}(\nu_1, \nu_2) e^{-\int_s^t K(r)\mathrm{d}r}.$$

(3) *When $p > 1$, for any $f \in \mathscr{B}_b^+(M)$ and $0 \leq s \leq t < T_c$,*

$$(P_{s,t} f)^p(x) \leq P_{s,t} f^p(y) \exp\left[\frac{p}{p-1} C(s, t, K)\rho_s^2(x, y)\right],$$

where $C(s, t, K) = \left[4 \int_s^t e^{2\int_s^r K(u)\mathrm{d}u} \mathrm{d}r\right]^{-1}$. And it keeps the same meaning in (4), (5), (6).

(4) *For any $f \in \mathscr{B}_b^+(M)$ with $f \geq 1$ and $0 \leq s \leq t < T_c$,*

$$P_{s,t} \log f(x) \leq \log P_{s,t} f(y) + C(s, t, K)\rho_s^2(x, y).$$

(5) *When $p > 1$, for any $0 \leq s \leq t < T_c$ and $x, y \in M$,*

$$\int_M p_{s,t}(x, y) \left(\frac{p_{s,t}(x, y)}{p_{s,t}(y, z)}\right)^{\frac{1}{p-1}} \mu_t(\mathrm{d}z) \leq \exp\left[\frac{p}{(p-1)^2} C(s, t, K)\rho_s^2(x, y)\right].$$

(6) *For any* $0 \le s \le t < T_c$ *and* $x, y \in M$,

$$\int_M p_{s,t}(x,y) \log \frac{p_{s,t}(x,y)}{p_{s,t}(y,z)} \mu_t(\mathrm{d}z) \le \rho_s^2(x,y) C(s,t,K).$$

(7) *For any* $0 \le s \le u \le t < T_c$ *and* $1 < q_1 \le q_2$ *such that*

$$\frac{q_2 - 1}{q_1 - 1} = \frac{\int_s^t e^{2\int_s^u K(r)\mathrm{d}r} \mathrm{d}u}{\int_s^u e^{2\int_s^u K(r)\mathrm{d}r} \mathrm{d}u} \tag{4.15}$$

there holds

$$\{P_{s,u}(P_{u,t}f)^{q_2}\}^{\frac{1}{q_2}} \le (P_{s,t} f^{q_1})^{\frac{1}{q_1}}, \ f \ge 0, f \in \mathscr{B}_b(M).$$

(8) *For any* $0 \le s \le u \le t < T_c$ *and* $0 < q_2 \le q_1$ *or* $q_2 \le q_1 < 0$ *such that* (4.15) *holds,*

$$(P_{s,t} f^{q_1})^{\frac{1}{q_1}} \le \{P_{s,u}(P_{u,t}f)^{q_2}\}^{\frac{1}{q_2}}.$$

(9) *For any* $0 \le s \le t < T_c$ *and* $f \in C_b^1(M)$,

$$|\nabla^s P_{s,t} f|_s^p \le e^{-p\int_s^t K(u)\mathrm{d}u} P_{s,t} |\nabla^t f|_t^p.$$

(10) *For any* $0 \le s \le t < T_c$ *and positive* $f \in C_b^1(M)$,

$$\frac{(p \wedge 2)\{P_{s,t} f^2 - (P_{s,t} f^{2/(p\wedge 2)})^{p\wedge 2}\}}{4(p \wedge 2 - 1)} \le \int_s^t e^{-2\int_u^t K(r)\mathrm{d}r} \mathrm{d}u P_{s,t} |\nabla^t f|_t^2.$$

When $p = 1$, *the inequality reduces to the log-Sobolev inequality*

$$P_{s,t}(f^2 \log f^2) - (P_{s,t} f^2) \log P_{s,t} f^2 \le 4 \int_s^t e^{-2\int_u^t K(r)\mathrm{d}r} \mathrm{d}u P_{s,t} |\nabla^t f|_t^2.$$

Proof. The equivalence of (1), (9) and (10) follows directly from Theorem 4.1 with continuous function K only dependent of time t. Moreover, according to the Young inequality, we see that (3) implies (4). Therefore, it remains to prove that (1) is equivalent to (2), (1) implies (3), (4) implies (1), and (10) with $p = 1$ is equivalent to each of (7) and (8).

(*a*) **(1) is equivalent to (2), (2′)**. By (1) and the index lemma, we have

$$\frac{1}{2} \int_\gamma \partial_t g_t(\dot\gamma(s), \dot\gamma(s)) \mathrm{d}s + I_t^Z(x,y) \le -K(t)\rho(x,y).$$

So, using the coupling by parallel displacement and Theorem 4.2 with $U = 0$, we obtain from (1) that

$$\begin{aligned} W_{p,t}(\delta_x P_{s,t}, \delta_y P_{s,t}) &\le \left\{ \mathbb{E}\left(\rho_t(X_t, \tilde X_t)^p \big| (X_s, \tilde X_s) = (x,y) \right) \right\}^{1/p} \\ &\le \rho_s(x,y) e^{-\int_s^t K(u)\mathrm{d}u}. \end{aligned}$$

That is, (1) implies (2). Obviously, (2′) implies (2). It is also easy to see that (2) implies (2′), so that they are equivalent. Indeed, let $\pi \in \mathscr{C}(\nu_1, \nu_2)$ such that $W_{p,s}(\nu_1, \nu_2) = \pi(\rho_s^p)^{1/p}$. Then from Monge-Kontorovich dual formula and (2) we obtain

$$\begin{aligned} W_{p,t}(\nu_1 P_{s,t}, \nu_2 P_{s,t})^p &\le \int_{M\times M} W_{p,s}(\delta_x P_{s,t}, \delta_y P_{s,t})^p \pi(\mathrm{d}x, \mathrm{d}y) \\ &\le e^{-p\int_s^t K(u)\mathrm{d}u} W_{p,s}(\nu_1, \nu_2)^p. \end{aligned} \tag{4.16}$$

On the other hand, if (2) holds that letting $\Pi_{x,y}$ be the optimal coupling for $\delta_x P_{s,t}$ and $\delta_y P_{s,t}$ for the L^p-transportation cost for $f \in C_b^1(M)$, we have

$$
\begin{aligned}
|\nabla^s P_{s,t} f|_s \le& \lim_{y\to x} \frac{\int_{M\times M} |f(x')-f(y')|\Pi_{x,y}(\mathrm{d}x',\mathrm{d}y')}{\rho_s(x,y)} \\
\le& \lim_{y\to x}\left[\int_{M\times M}\left(\frac{|f(x')-f(y')|}{\rho_t(x',y')}\right)^{p/(p-1)}\Pi_{x,y}(\mathrm{d}x',\mathrm{d}y')\right]^{(p-1)/p} \cdot \frac{W_{p,t}(\delta_x P_{s,t},\delta_y P_{s,t})}{\rho_s(x,y)} \\
\le& e^{-\int_s^t K(u)\mathrm{d}u}\left(P_{s,t}|\nabla^t f|_t^{p/(p-1)}\right)^{(p-1)/p}.
\end{aligned}
$$

By Theorem 4.1, this implies (1).

(*b*) (**1**) **implies** (**3**). We also consider the case for $s=0$. By approximation and the monotone class theorem, we may assume that $f \in C_b^2(M), \inf f > 0$ and f is constant outside a compact set. Given $x \ne y$ and $t>0$, let $\gamma : [0,t] \to M$ be the g_0-geodesic from x to y with length $\rho_0(x,y)$. Let $\nu_s = \frac{\mathrm{d}\gamma_s}{\mathrm{d}s}$, we have $|\nu_s|_0 = \rho_0(x,y)/t$. Let

$$
h(s) = \frac{t\int_0^s e^{2\int_0^r K(u)\mathrm{d}u}\mathrm{d}r}{\int_0^t e^{2\int_0^r K(u)\mathrm{d}u}\mathrm{d}r}.
$$

Then $h(0)=0$, $h(t)=t$. Let $y_s = \gamma_{h(s)}$. Define

$$
\varphi(s) = \log P_{0,s}(P_{s,t}f)^p(y_s),\ \ s\in[0,t].
$$

By $|\nabla^0 P_{0,t} f|_0 \le e^{-\int_0^t K(s)\mathrm{d}s} P_{0,t}|\nabla^t f|_t$ implied by (1) according to Theorem 4.1, and using the Kolmogrov equations, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}\varphi(s)}{\mathrm{d}s} =& \frac{1}{P_{0,s}(P_{s,t}f)^p}\Big\{P_{0,s}L_s(P_{s,t}f)^p(y_s) - pP_{0,s}(P_{s,t}f)^{p-1}L_sP_{s,t}f(y_s) \\
&+ h'(s)\left\langle \nabla^0 P_{0,s}(P_{s,t}f)^p, \nu_s\right\rangle_0\Big\} \\
\ge& \frac{p}{P_{0,s}(P_{s,t}f)^p}\Big\{p(p-1)P_{0,s}(P_{s,t}f)^{p-2}|\nabla^s P_{s,t}f|_s^2 \\
&- \frac{\rho_0(x,y)}{t}e^{-\int_0^s K(u)\mathrm{d}u}h'(s)(P_{s,t}f)^{p-1}|\nabla^s P_{s,t}f|_s\Big\} \\
=& \frac{p}{P_{0,s}(P_{s,t}f)^p}P_{0,s}\Big\{(P_{s,t}f)^p\Big((p-1)|\nabla^s \log P_{s,t}f|_s^2 \\
&- \frac{\rho_0(x,y)}{t}h'(s)e^{-\int_0^s K(u)\mathrm{d}u}|\nabla^s \log P_{s,t}f|_s\Big)\Big\} \\
\ge& \frac{-p\rho^2 h'(s)^2 e^{-2\int_0^s K(u)\mathrm{d}u}}{4(p-1)t^2},\ \ s\in[0,t].
\end{aligned}
$$

Since $h'(s) = \frac{te^{2\int_0^s 2K(u)\mathrm{d}u}}{\int_0^t e^{\int_0^r K(u)\mathrm{d}u}\mathrm{d}r}$, we have

$$
\frac{\mathrm{d}\varphi(s)}{\mathrm{d}s} \ge \frac{-p\rho_0(x,y)^2 e^{\int_0^s 2K(u)\mathrm{d}u}}{4(p-1)(\int_0^t e^{2\int_0^z K(u)\mathrm{d}u}\mathrm{d}z)^2},\ \ s\in[0,t].
$$

By integrating over s from 0 and t, we complete the proof.

(*c*) (**4**) **implies** (**1**). Let $x \in M$ and $X \in T_xM$ be fixed. For any $n \ge 1$ we may take $f \in C^\infty(M)$ such that $f \ge n$, f is constant outside a compact set, and

$$
\nabla^0 f(x) = X,\ \ \mathrm{Hess}_f^0(x) = 0.
$$

Taking $\gamma_t = \exp_x\left[-2t\nabla^0 \log f(x)\right]$, we have $\rho_0(x,\gamma_t) = 2t|\nabla^0 \log f|_0(x)$ for $t \in [0,t_0]$, where $t_0 > 0$ is such that $\rho_0(x,\gamma_t) < r$, $r > 0$. By (4) with $y = \gamma_t$, we obtain

$$P_{0,t}(\log f)(x) \le \log P_{0,t} f(\gamma_t) + \frac{t^2|\nabla^0 \log f|_0^2(x)}{\int_0^t e^{2\int_0^r K(u)\mathrm{d}u}\mathrm{d}r}, \quad t \in [0,t_0]. \tag{4.17}$$

Since $L_0 f \in C_0^2(M)$ and $\mathrm{Hess}_f^0(x) = 0$ implies $\nabla^0|\nabla^0 f|_0^2(x) = 0$ at point x, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} P_{0,t}\log f &= L_0 \log f = \frac{L_0 f}{f} - \frac{1}{f^2}|\nabla^0 f|_0^2;\\
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} P_{0,t}\log f &= \frac{\mathrm{d}}{\mathrm{d}t}(P_{0,t}L_t \log f)|_{t=0}\\
&= \frac{L^2(0)f}{f} - \frac{(L_0 f)^2}{f^2} - \frac{2}{f^2}\left\langle \nabla^0 L_0 f, \nabla^0 f\right\rangle_0 - \frac{L_0|\nabla^0 f|_0^2}{f^2}\\
&\quad + \frac{4|\nabla^0 f|_0^2 L_0 f}{f^3} - \frac{6|\nabla^0 f|_0^4}{f^4} + \frac{\partial_t g_t|_{t=0}(\nabla^0 f, \nabla^0 f)}{f^2} + \frac{1}{f}\frac{\mathrm{d}L_t f}{\mathrm{d}t}\Big|_{t=0}\\
&:= A.
\end{aligned}$$

Thus, by Taylor's expansions,

$$P_{0,t}(\log f)(x) = \log f(x) + t(f^{-1}L_0 f - |\nabla^0 \log f|_0^2)(x) + \frac{t^2}{2}A + o(t^2) \tag{4.18}$$

holds for small $t > 0$. On the other hand, let $N_t = P^0_{x,\gamma_t}\nabla^0 \log f(x)$, where P^0_{x,γ_t} is the g_0-parallel displacement along the g_0-geodesic $t \to \gamma_t$. We have $\dot\gamma_t = -2N_t$ and $\nabla^0_{\dot\gamma_t} N_t = 0$. So,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \log P_{0,t} f(\gamma_t) &= \left(\frac{P_{0,t}L_t f(\gamma_t)}{P_{0,t}f} + \frac{\left\langle \nabla^0 P_{0,t} f, \dot\gamma_t\right\rangle_0}{P_{0,t} f}(\gamma_t)\right)\Big|_{t=0}\\
&= \frac{L_0 f}{f} - 2|\nabla^0 \log f|_0^2,\\
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} \log P_{0,t} f(\gamma_t) &= \frac{L_0^2 f}{f} - \frac{1}{f^2}(L_0 f)^2 + \frac{2}{f^2}L_0 f\left\langle \nabla^0 f, \nabla^0 \log f\right\rangle_0\\
&\quad + 4\mathrm{Hess}^0_{\log f}(\nabla^0 f, \nabla^0 f) - \frac{2}{f}\left\langle \nabla^0 L_0 f, \nabla^0 \log f\right\rangle_0\\
&\quad - 2\left\langle f^{-1}\nabla^0 L_0 f, \nabla^0 \log f\right\rangle_0 + \frac{1}{f}\frac{\mathrm{d}L_t f}{\mathrm{d}t}\Big|_{t=0}.
\end{aligned}$$

where, as in above, the functions take value at point x and we have used $\mathrm{Hess}_f^0(x) = 0$ in the last step. Thus, we have

$$\log P_{0,t} f(\gamma_t) = \log f(x) + t(f^{-1}L_0 f - 2|\nabla^0 \log f|_0^2)(x) + \frac{t^2}{2}B + o(t^2).$$

Combining this with (4.17) and (4.18), we arrive at

$$\begin{aligned}
&\frac{1}{t}\left(1 - \frac{t}{4\int_0^t e^{2\int_0^r K(u)\mathrm{d}u}\mathrm{d}r}\right)|\nabla^0 \log f|_0^2(x)\\
\le&\frac{1}{2}\Bigg(\frac{L_0|\nabla^0 f|_0^2 - 2\left\langle \nabla^0 L_0 f, \nabla^0 f\right\rangle_0}{f^2} + \frac{2|\nabla^0 f|_0^4}{f^4}\\
&+ \frac{1}{f^2}\partial_t g_t|_{t=0}(\nabla^0 f, \nabla^0 f)\Bigg)(x) + o(1)
\end{aligned}$$

Letting $t \to 0$, we obtain

$$\begin{aligned}\Gamma_2^0(f,f)(x) :=& \frac{1}{2}L_0|\nabla^0 f|_0^2(x) - \left\langle \nabla^0 L_0 f, \nabla f\right\rangle_0(x)\\ \geq& K(0)|\nabla^0 f|_0^2(x) + \frac{1}{2}\partial_t g_t|_{t=0}(\nabla^0 f, \nabla^0 f)(x) - \frac{|\nabla^0 f|_0^4}{f^2}(x). \end{aligned}\tag{4.19}$$

Since by the Bochner-Weitzenböck formula and $\nabla^0 f(x) = X,\ f(x) \geq n$ and

$$\Gamma_2^0(f,f)(x) = \mathrm{Ric}^0(X,X) - \left\langle \nabla_X^0 Z_0, X\right\rangle_0,$$

it follows that

$$\mathrm{Ric}^0(X,X) - \left\langle \nabla_X^0 Z_0, X\right\rangle_0 \geq K(0)|X|_0^2 + \frac{1}{2}\partial_t g_t|_{t=0}(X,X) - \frac{|X|_0^4}{n},\ n \geq 1.$$

This implies (1) by letting $n \to \infty$.

(*d*) (**10**) **with** $p = 1$ **implies** (**7**) **and** (**8**). We again prove this assertion for $s = 0$. By an approximation argument, it suffices to prove for $f \in C_b^\infty(M)$ such that $\inf f > 0$ and $L_t f$ is bounded. In this case, for any $t > 0$, let

$$q(s) = 1 + \frac{(q_2 - 1)\int_0^t e^{2\int_0^r K(u)\mathrm{d}u}\mathrm{d}r}{\int_0^s e^{2\int_0^r K(u)\mathrm{d}u}\mathrm{d}r},\ \psi(s) = \{P_{0,s}(P_{s,t}f)^{q(s)}\}^{\frac{1}{q(s)}},\ s \in (0,t].$$

Then

$$\int_0^s e^{-2\int_r^s K(u)\mathrm{d}u}\mathrm{d}r + \frac{q(s)-1}{q'(s)} = 0.$$

So that (10) with $p = 1$ implies

$$\begin{aligned}\left(\frac{\psi'\psi^{q-1}q^2}{q'}\right)(s) =& P_{0,s}(P_{s,t}f)^{q(s)}\log(P_{s,t}f)^{q(s)} - \psi(s)^{q(s)}\log\psi(s)\\ &+ \frac{q(s)}{q'(s)}\Big[P_{0,s}L_s(P_{s,t}f)^{q(s)} - q(s)P_{0,s}(P_{s,t}f)^{q(s)}L_sP_{s,t}f\Big]\\ =& P_{0,s}(P_{s,t}f)^{q(s)}\log(P_{s,t}f)^{q(s)} - P_{0,s}(P_{s,t}f)^{q(s)}\log P_{0,s}(P_{s,t}f)^{q(s)}\\ &+ \frac{q(s)^2(q(s)-1)}{q'(s)}P_{0,s}(P_{s,t}f)^{q(s)-2}|\nabla^s p_{s,t}f|_s^2\\ \leq& q(s)^2\left(\int_0^s e^{-2\int_u^t K(r)\mathrm{d}r}\mathrm{d}u + \frac{q(s)-1}{q'(s)}\right)P_{0,s}(P_{s,t}f)^{q(s)-2}|\nabla^s P_{s,t}f|_s^2 = 0.\end{aligned}$$

Therefore, in case (7) one has $q'(s) < 0$ so that $\psi'(s) \geq 0$, while in case (8) one has $q'(s) < 0$ so that $\psi'(s) \leq 0$. Hence, the inequalities in (7) and (8) hold.

(*e*) (**7**) **or** (**8**) **implies** (**10**) **with** $p = 1$. We only prove that (7) implies (10), since (8) implying (10) can be proved in a similar way. Let $q_1 = 2$ and $q_2 = 2(1+\varepsilon)$ for small $\varepsilon > 0$. According to (4.15) we take

$$\frac{1}{1+2\varepsilon} = \frac{\int_0^s e^{2\int_0^r K(u)\mathrm{d}u}\mathrm{d}r}{\int_0^t e^{2\int_0^r K(u)\mathrm{d}u}\mathrm{d}r} = 1 - \frac{\int_s^t e^{2\int_0^r K(u)\mathrm{d}u}\mathrm{d}r}{\int_0^t e^{2\int_0^r K(u)\mathrm{d}u}\mathrm{d}r}.$$

Then,

$$\frac{2\varepsilon}{1+2\varepsilon}\int_0^t e^{2\int_0^r K(u)\mathrm{d}u}\mathrm{d}r = \int_s^t e^{2\int_0^r K(u)\mathrm{d}u}\mathrm{d}r \sim (t-s)e^{2\int_0^t K(u)\mathrm{d}u}.$$

i.e.

$$t - s \sim 2\varepsilon \int_0^t e^{2\int_r^t K(u)\mathrm{d}u} \mathrm{d}r.$$

Denote $\theta = 2\int_0^t e^{2\int_r^t K(u)\mathrm{d}u} \mathrm{d}r$. We obtain from (7) that

$$\begin{aligned} 0 &\geq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \{ (P_{0,t-\theta\varepsilon}(P_{t-\theta\varepsilon,t})^{2(1+\varepsilon)})^{\frac{1}{2(1+\varepsilon)}} - (P_{0,t}f^2)^{1/2} \} \\ &= P_{0,t}f^2 \log f^2 - (P_{0,t}f^2)\log P_{0,t}f^2 - 4\int_0^t e^{-2\int_u^t K(r)\mathrm{d}r} \mathrm{d}u P_{0,t}|\nabla^0 f|_0^2. \end{aligned}$$

Therefore, (10) with $p = 1$ holds. □

When the metric is independent of t, the equivalence of (1) and (2) is due to [24], (3) was initiated in [27] while the equivalent of (1) and (4) are essentially due to [28], and (7)-(8) are found in [7].

Remark 4.5. *According to [33, Propersition 2.4], we have the following statements are equivalent to Theorem 4.4 (3), (4) respectively.*

(3′) *For any $0 \leq s \leq t < T_c$, $p_{x,y}^{s,t}$ satisfies*

$$P_{s,t}((p_{x,y}^{s,t})^{1/(\alpha-1)})(x) \leq \left\{ \frac{p}{p-1} C(s,t,K)\rho_s^2(x,y) \right\}^{1/(\alpha-1)}, \quad x, y \in E. \tag{4.20}$$

(4′) *For any $0 \leq s \leq t < T_c$, $p_{x,y}^{s,t}$ satisfies*

$$P_{s,t}\{\log p_{x,y}^{s,t}\}(x) \leq C(s,t,K)\rho_s^2(x,y), \quad x, y \in E.$$

where $C(s,t,K) = \left[4\int_s^t e^{2\int_s^r K(u)\mathrm{d}u} \mathrm{d}r \right]^{-1}$.

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